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On Some Open Problems of Approximation Theory*

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> *In re mathematica ars proponendi questionem pluris facienda est, quam solvendi.* [*In mathematics, the art of formulating a problem is more valuable than that of solving it.*]
>
> G. Cantor

1. The present paper contains (with some additions) part of my lectures held in Summer 1975 at the Université de Montreal. It does not contain new results, with the exception, perhaps, of §73. It is mainly a systematic exposition of some open problems, to which I was led by working in the field for a long time. The problems are of various degrees of difficulty but are not arranged in that order. I shall indicate the problems which did not originate with me. The most frequently mentioned name will be P. Erdös, who initiated the genre "problem-paper" and who has been working with me for many years in approximation theory. Even if all or some of the problems treated do not satisfy the above-quoted maxim of Cantor, I still think that most of them are problems worthy of study.

I. LAGRANGE INTERPOLATION

2. Perhaps it would be interesting to dig to the roots of the theory and to indicate its historical origins. Newton, who wanted to draw conclusions from the observed location of comets at equidistant times as to their location at arbitrary times arrived at the problem of determining a "geometric" curve passing through arbitrarily many given points. He solved this problem by the interpolation polynomial bearing his name. How highly he esteemed his result is revealed by his letter to Oldenburg of 1676, in which he wrote that this was one of the most beautiful results he had ever achieved. Newton uses his formula to give the exact value of $\int_a^b f(x)\,dx$ in terms of the values

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of $f(x_\nu)$ when $f(x)$ is a polynomial of degree n, and $x_\nu = a + ((b-a)/n)\,\nu$ ($\nu = 0,\ldots, n$). His student Cotes called this quadrature formula "pulcherrima et utilissima regula" and calculated its coefficients for $n \leqslant 10$. This work, based on Newton's interpolation formula, must have bee quite awkward. Application of Lagrange's interpolation formula would have simplified it, but that was published only in 1795. Gauss's quadrature formula was also motivated by astronomy, namely by the investigation of the orbit of the planet Pallas. How important this formula was for Gauss is shown by the fact that unlike many other results, this one was not only worked out in his diary but was also published, even prepublished. The essential novelty, compared to Newton–Cotes's formula, was that Gauss used the zeros of the nth Legendre polynomial instead of equidistant points of observation. His treatment was later greatly simplified by Jacobi.

Thus we see that interpolation and the theory of mechanical quadrature are just two aspects of the study of functions given by a finite number of observations.

3. Because of the notion of a function of that time, it was generally believed that Newton–Cotes' quadrature formula as well as that of Gauss converge to the integral of $f(x)$ as $n \to \infty$. Only toward the end of the last century was it noticed by Borel and Runge that in $[-1, 1]$ (which is no restriction of generality), for the quadrature formula using equidistant points, even such a function as $(1 + x^2)^{-1}$ can be "bad." The Newton–Cotes procedure can diverge even for functions analytic in a domain containing the interval $[-1, 1]$. This was proved by Pólya in 1933.

4. The question of convergence of Gauss's formula was raised by Chebyshev who conjectured an affirmative answer to it in 1874. His conjecture was proved 10 years later by Stieltjes and A. Markov, independently. In fact, they found that for the convergence of Gauss's quadrature procedure, Riemann-integrability of the function is sufficient. After this discovery, the question naturally arose whether by replacing equidistant points by the zeros of the nth Legendre polynomial the behavior of Lagrange interpolation could be improved. It took another 30 years until this question was settled. After the theorem of Stieltjes and Markov and the approximation theorem of Weierstrass, it was hoped that there exists a (non-equidistant) system of nodes for which Lagrange's interpolation polynomials converge uniformly, for every function continuous in $[-1, 1]$. The mathematical world was awakened from this dream in 1914 by Faber [17] who showed that there is *no* such a system.

We explain at this point our notation to be used later.

Let

$$A\colon 1 \geqslant x_{1n} > x_{2n} \cdots > x_{nn} \geqslant -1 \tag{4.1}$$

be the basic points of interpolation. These points form an infinite triangular matrix. The corresponding Lagrange interpolation polynomial is denoted by $L_n(x, f, A)$, $L_n(f, A)$ or $L_n(f)$. We have

$$L_n(x, f; A) = \sum_{\nu=1}^{n} f(x_{\nu n})\, l_{\nu n}(x, A) = \sum_{\nu=1}^{n} f(x_\nu)\, l_\nu(x, A) = \sum_{\nu=1}^{n} f(x_\nu)\, l_\nu(x), \quad (4.2)$$

where

$$l_{\nu n}(x) = \frac{\omega_n(x; A)}{\omega_n'(x_{\nu n}; A)(x - x_{\nu n})} = \frac{\omega_n(x)}{\omega_n'(x_\nu)(x - x_\nu)}, \quad (4.3)$$

$$\omega_n(x, A) = \prod_{\nu=1}^{n} (x - x_{\nu, n}) = \prod_{\nu=1}^{n} (x - x_\nu). \quad (4.4)$$

Faber showed in 1914, much before the theorem of Banach–Steinhaus, but after the constructions of Lebesgue and Haar, that for the uniform convergence of the Lagrange polynomials for every function continuous on $[-1, 1]$ it is necessary that

$$M_n(A) = \max_{-1 \leqslant x \leqslant 1} \sum_{\nu=1}^{n} |l_{\nu, n}(x)| \leqslant C \quad (4.5)$$

with some constant C independent of n. On the other hand, he showed that, for every matrix A,

$$M_n(A) > C_1 \log n. \quad (4.6)$$

Hence (4.5) can never be true.

5. Before we proceed, I would like to mention a particularly important class of matrices A.

Let

$$p(x) \geqslant 0, \qquad p(x) \in L(-1, +1). \quad (5.1)$$

As is well known, there exists a uniquely determined (up to constant factors) system of polynomials

$$q_0(x), q_1(x), \ldots \quad (5.2)$$

for which

$$\int_{-1}^{1} q_n(x)\, q_\nu(x)\, p(x)\, dx = 0 \qquad \text{for } n \neq \nu. \quad (5.3)$$

These polynomials q_n are called orthogonal polynomials with weight p. It is well known that the zeros of these polynomials are simple and lie in

$(-1, 1)$. A special class of matrices A is the class where the nth row consists of the zeros of $q_n(x)$. Such matrices are called p-matrices. Of special impotance are the matrices belonging to the weight function

$$p(x) = (1 - x)^\alpha(1 + x)^\beta \qquad (\alpha > -1, \beta > -1). \tag{5.4}$$

The polynomials $q_0(x), q_1(x),...$ are the Jacobi polynomials belonging to the parameters α, β; they are denoted $P_n^{(\alpha,\beta)}(x)$. Their importance is motivated by the fact that $P_n^{(0,0)}(x)$ is the nth Legendre polynomial mentioned in §2 and $P_n^{(-1/2,-1/2)}(x)$ is the nth Chebychev polynomial $T_n(x)$ satisfying

$$T_n(\cos\theta) = c\cos n\theta \qquad (c \text{ const}). \tag{5.5}$$

The p-matrix belonging to $(1 - x)^\alpha(1 + x)^\beta$ will be denoted

$$P(\alpha, \beta). \tag{5.6}$$

The Laguerre polynomials $L_n^*(t)$ and the Hermite polynomials $K_n(t)$ play an important role in the theory. They are defined by

$$\int_0^\infty L_n^*(t)\, t^\nu e^{-t}\, dt = 0$$

and

$$\int_{-\infty}^\infty K_n(t)\, t^\nu e^{-t^2}\, dt = 0, \qquad \nu = 0, 1,..., n - 1; n = 1, 2,...,$$

respectively.

6. To motivate our first problem, we start with the following question. Let the function $f(x)$ be known merely by observations at the points

$$1 \geqslant x_1 > \cdots > x_n \geqslant -1. \tag{6.1}$$

We want to calculate it (approximately) at an arbitrary point x of $[-1, +1]$ as

$$L_n(f) = \sum_{\nu=1}^n f(x_\nu)\, l_\nu(x). \tag{6.2}$$

We would like to diminish the effect of the errors of observation. If $f^*(x_\nu)$ is the "true" value of $f(x)$ at $x = x_\nu$, and

$$\max_{1\leqslant\nu\leqslant n} |f(x_\nu) - f^*(x_\nu)| = \delta, \tag{6.3}$$

then the most we can say is that the effect of the errors of observation does not exceed

$$\delta \max_{-1 \leqslant x \leqslant 1} \sum_{\nu=1}^{n} |l_\nu(x)| . \tag{6.4}$$

We would like to select the most favorable points of observation. If we are able to choose these points so that

$$\max_{-1 \leqslant x \leqslant 1} \sum_{\nu=1}^{n} |l_\nu(x)|$$

is minimal, then we have found the Lagrange interpolation which is "least sensitive" to observation errors. We call this interpolation the "most stable" one for n observations.

Using the notation of §4, we state the following (well-known)

PROBLEM I. *What are the matrices A for which*

$$M_n(A) = \max_{-1 \leqslant x \leqslant 1} \left| \sum_{\nu=1}^{n} l_\nu(x) \right| \tag{6.5}$$

is minimal?

The question is settled only for $n \leqslant 4$. One of the last papers about this subject is that of F. Schurer (*Studia Sci. Math. Hungar.* 1974). It was conjectured for a long time that the extremal matrix is $P(-\frac{1}{2}, -\frac{1}{2})$. For small values of n this is false. On the other hand, it is true and known that, denoting the Chebyshev matrix $P(-\frac{1}{2}, -\frac{1}{2})$ by T, one has

$$\left| M_n(T) - \frac{2}{\pi} \log n \right| \leqslant c_2 , \tag{6.6}$$

c_2 being a constant.

7. Relations (4.6) and (6.7) show that, essentially, Faber's theorem cannot be improved. But from the point of view of stability, even a multiplicative constant is important. Therefore, for Erdös and me it was worthwhile to investigate the asymptotic behavior of $M_n(A)$. In [15] we showed that

$$M_n(A) \geqslant \frac{2}{\pi} \log n - c_3 \log \log n. \tag{7.1}$$

Using a more difficult argument, Erdös later showed that

$$M_n(A) \geqslant \frac{2}{\pi} \log n - c_4 \tag{7.2}$$

for all matrices A.

From this and (6.7) it follows that

$$\lim_{n\to\infty}\left(\frac{1}{\log n}\min_A M_n(A)\right)=\frac{2}{\pi}. \tag{7.3}$$

Moreover, for $n \geqslant 2$,

$$\left|\min_A M_n(A)-\frac{2}{\pi}\log n\right|<c_5. \tag{7.4}$$

8. There is another important application of (6.7) where the value of the multiplicative constant is unimportant. Namely, it is easy to see that if

$$f(x_1)-f(x_2)=\frac{o(1)}{\log\dfrac{1}{|x_1-x_2|}} \tag{8.1}$$

whenever $1 \geqq x_1 > x_2 \geqq -1$, then we have

$$L_n(f, T)\to f(x), \tag{8.2}$$

uniformly in $[-1, 1]$.

Similarly, if

$$M_n(A)<c_7\log n, \tag{8.3}$$

or

$$\max_{-1+\delta\leqslant x\leqslant 1-\delta}\sum_{\nu=1}^{n}|l_{\nu,n}(x)|<c_8(\delta)\log n, \tag{8.4}$$

and if (8.1) is satisfied, then the uniform convergence (8.2) holds on $[-1, +1]$, or, respectively, on $[-1+\delta, 1-\delta]$. In his monograph "Orthogonal Polynomials," G. Szegö showed that (8.4) holds for any $P(\alpha, \beta)$-matrix. However, it seems to be very difficult to answer

PROBLEM II. *Is* (8.4) *true for every p-matrix* (*see* §5) *if* $c_8(\delta)$ *is replaced by* $c_8(\delta, p)$ *and if, on* $[-1, +1]$; *we have*

$$p(x)\geqslant c>0?$$

Under an assumption on $p(x)$ which cannot easily be verified, (8.4) was proved by Freud [23]. On the other hand, I showed with Grünwald in 1938 [30] that, if

$$p(x)\geqslant\frac{c_9}{(1-x^2)^{1/2}}, \tag{8.5}$$

then, for the corresponding p-matrix P,

$$\max_{-1\leqslant x\leqslant +1} \sum |l_{\nu n}(x; P)| < c_{10}\,(n)^{1/2}.$$

9. Faber's theorem asserts only that for every matrix A there exists a continuous function $f_0(x)$ such that its Lagrange interpolation polynomials do not converge *uniformly* to $f_0(x)$. Thus, it would still be possible that for some matrix A, its Lagrange interpolating polynomials for some continuous function $f(x)$ converge to $f(x)$ for every $x \in [-1, 1]$. Even this is false, as Bernstein [5] proved in 1931. He showed that, for each matrix A, there is a function $f_1(x) \in C[-1, +1]$, and a point in $[-1, +1]$ for which the values of $L_n(f_1 ; A)$ are unbounded as $n \to \infty$. The proof is easily accomplished by strenghtening the result (4.6) of Faber to

$$\max_{a\leqslant x\leqslant b} \sum_{\nu=1}^{n} |l_{\nu,n}(x, A)| > c(a, b) \log n, \tag{9.1}$$

where $[a, b]$ is an arbitrarily small subinterval of $[-1, +1]$. Actually, for this purpose, it suffices to extablish that

$$\overline{\lim_{n\to\infty}} \max_{a\leqslant x\leqslant b} \sum_{\nu=1}^{n} |l_{\nu,n}(x, A)| = \infty. \tag{9.2}$$

Still stronger phenomena of divergence were discovered in 1935 by G. Grünwald [28, 29] and (independently) by Marcinkiewicz [37] in the case of the T-matrix which can be considered as the "best" one. They showed the existence of a continuous $f_2(x)$ such that $L_n(f_2 , T)$ is unbounded *everywhere* in $[-1, +1]$ as $n \to \infty$. To prove this, in addition to many deep ideas, it was also necessary to show that $\sum_{\nu=1}^{n} | l_{\nu,n}(x, T)|$ is unbounded as $n \to \infty$.

Erdös [14] proved in 1958 that, for every matrix A, we have

$$\overline{\lim_{n\to\infty}} \sum_{\nu=1}^{n} | l_{\nu,n}(x, A)| = \infty \qquad \text{almost everywhere.} \tag{9.3}$$

The following question is still open:

PROBLEM III (P. Erdös). *Does there exist, for every A, a function $f_3 \in C[-1, +1]$ with the property that*

$$\overline{\lim_{n\to\infty}} | L_n(f_3 , A)| = \infty \tag{9.4}$$

for all $x \in [-1, 1]$ except possibly for a set of measure zero?

The fact that the answer is negative if "almost everywhere" is replaced by "everywhere" is shown by special matrices of the form

$$\begin{array}{l} x_1 \\ x_1, x_2 \\ x_1, x_2, x_3 \\ \cdots\cdots \end{array}$$

There is another aspect which makes the theorem of Grünwald–Marcinkiewicz very interesting. It is easy to see that $L_n(f, T)$ can be written as

$$a_0 + \sum_{r=1}^{n-1} a_r \cos r\theta,$$

where

$$a_0 = \frac{1}{n} \sum_{k=1}^{n} f\left(\cos \frac{2k-1}{2n} \pi\right)$$

and, for $r \geqslant 1$,

$$a_r = \frac{2}{n} \sum_{k=1}^{n} f\left(\cos \frac{2k-1}{2n} \pi\right) \cos r \frac{2k-1}{2n} \pi.$$

This is similar to the $(n - 1)$th partial sum of the cosine Fourier series of $f(\cos \theta)$. That theorem could be a basis for the conjecture that the Fourier series of a continuous periodic function can be everywhere divergent, and according to the theorem of Carleson, this is false.

10. In the introduction to our paper [9], Erdös and I were very cautious in making predictions about the possibility of convergence of $L_n(f, A)$ at $x = x_0$ to a value different from $f(0)$ for some matrix A. At the end of the paper, however, we made three remarks. First, as was shown by Marcinkiewicz, with the notation (5.6), that for $A = P(\frac{1}{2}, \frac{1}{2})$, the Lagrange interpolation polynomials at a point x cannot converge to anything but the value of the function there. Second, for $T = P(-\frac{1}{2}, -\frac{1}{2})$, the same is true if

$$x_0 \neq \cos \frac{l\pi}{k}, \qquad (k, l) = 1,\ k \text{ and } l \text{ odd}. \tag{10.1}$$

Finally, for $x_0 = \cos(\pi/3)$, $L_n(x_0, f_0, T)$ can converge to any given value, even to ∞, for a suitable $f_0(x) \in C[-1, +1]$.

The following two problems arise now in a natural way.

PROBLEM IV. *What are the properties of the set of points x* not *satisfying* (10.1) *for which the Lagrange interpolation polynomials $L_n(f_1, T)$ can converge to values different from $f_1(x)$?*

PROBLEM V. *How "large" is the subset of points x_0 of* $[-1, +1]$ *for which $L_n(f_2, A)$ can converge to a value different from $f_2(x_0)$ with a given A and an appropriately chosen $f_2(x)$?*

11. In addition to (9.3), Erdös [14] makes also the stronger assertion that, for arbitrarily small $\epsilon > 0$, and sufficiently small $\eta = \eta(\epsilon)$, we have

$$\sum_{\nu=1}^{n} | l_{\nu,n}(x, A)| > \eta \log n \tag{11.1}$$

for all $x \in [-1, 1]$, except, possibly, for a set of measure not exceeding ϵ. Instead of proving (11.1) he only remarks that the proof is analogous to that of (9.3) but more complicated. So we have

PROBLEM VI (P. Erdös). *Work out the proof of* (11.1).

From (11.1) is would follow that

$$\int_{-1}^{1} \sum_{\nu=1}^{n} | l_{\nu n}(x, A) | \, dx > 2\eta \log n. \tag{11.2}$$

Instead of this, the followig extremum problem could probably be solved directly.

PROBLEM VII. *Determine the matrices A which minimalize the integral*

$$\int_{-1}^{1} \sum_{\nu=1}^{n} | l_{\nu n}(x, A)| \, dx.$$

12. We return to the subject of mechanical quadratures. In §4 I have already mentioned that the situation here is not as bad as in the case of Lagrange interpolation. We have to investigate the behavior of

$$Q_n(f) = \sum_{\nu=1}^{n} f(x_{\nu n}) \int_{-1}^{1} l_{\nu n}(x, A) \, dx, \tag{12.1}$$

where A is a given matrix. We can restrict ourselves to functions defined in $[-1, -1]$ and belonging to some fixed class of functions. Define the Cotes numbers as

$$\int_{-1}^{1} l_{\nu n}(x, A) \, dx = \lambda_{\nu n}(A) = \lambda_{\nu n}, \qquad \nu = 1, 2, \ldots, n;\ n = 1, 2, \ldots. \tag{12.2}$$

A necessary and sufficient condition for the equality

$$\lim_{n\to\infty} Q_n(f) = \int_{-1}^{1} f(x)\,dx \tag{12.3}$$

to hold for all $f \in C[-1, 1]$ was found in 1918 by Hahn [32]. This, again, predates Banach and Steinhaus's celebrated theorem, and also Hahn's own 85-page paper in *Monatshefte für Math. und Physik* (1922), where these questions were treated in abstract form. The condition is

$$\Delta_n(A) \overset{\text{def}}{=} \sum_{\nu=1}^{n} |\lambda_\nu(A)| < C, \tag{12.4}$$

with C independent of n.

Although, as already mentioned, the first theorem guaranteeing convergence was proved in the last century (for the Legendre matrix $P(0, 0)$), the first general theorem on this subject was proven by Erdös and myself [9] in 1934. This theorem asserts that, for every p-matrix P, and every Riemann-integrable function f, we have

$$\lim_{n\to\infty} \int_{-1}^{1} (f(x) - L_n(f, P))^2\, p(x)\,dx = 0, \tag{12.5}$$

An important special case is when

$$p(x) \geqslant c > 0; \tag{12.6}$$

then

$$\lim_{n\to\infty} \int_{-1}^{1} (f(x) - L_n(f, P))^2\,dx = 0; \tag{12.7}$$

if, instead of (12.6), we assume the weaker condition

$$\frac{1}{p(x)} \in L(-1, +1), \tag{12.8}$$

then

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - L_n(f, P)|\,dx = 0, \tag{12.9}$$

a result which was new even for the Markov–Stieltjes case $P(0, 0)$. Our next problem deals with the question whether or not the exponent 2 in (12.5) can be improved. More exactly,

PROBLEM VIII. *Does there exist a p_0-matrix P_0 such that, for some $f_0 \in C[-1, +1]$, we have*

$$\lim_{n\to\infty} \int_{-1}^{1} |f_0(x) - L_n(f_0, P_0)|^\lambda\, p_0(x)\,dx = \infty$$

for every $\lambda > 2$?

A somewhat weaker form of this problem would be

PROBLEM IX. *Does there exist a p_1-matrix P_1 such that, for every given $\lambda > 2$, there is an $f_2 \in C[-1, +1]$ with*

$$\overline{\lim_{n\to\infty}} \int_{-1}^{1} |f_2(x) - L_n(f_2, P_1)|^\lambda p_1(x)\, dx = \infty?$$

As orientation I would like to mention a theorem of Askey [1], according to which, for every given $\lambda > 2$, there is a weight function $p_2(x)$ such that, with an appropriate $f_2 \in C[-1, +1]$, for the p_2-matrix P_2, we have

$$\overline{\lim_{n\to\infty}} \int_{-1}^{1} |f_2 - L_n(f_2, P_2)|^\lambda p_2(x)\, dx = \infty.$$

13. For more special weight functions $p(x)$, one can expect the validity of a stronger theorem than 12.5. In fact, Erdös and Feldheim [8] proved in 1936 that, for $P = T$, and for arbitrarily large integers k, we have

$$\lim_{n\to\infty} \int_{-1}^{1} (f(x) - L_n(f, T))^{2k} \frac{dx}{(1 - x^2)^{1/2}} = 0, \tag{13.1}$$

whenever $f \in C[-1, 1]$.

Hence we pose

PROBLEM X (Erdös–Feldheim). *Is it true that, for every $k > 0$,*

$$\lim_{n\to\infty} \int_{-1}^{1} (f - L_n(f, P))^{2k}\, dx = 0 \tag{13.2}$$

if $f \in C[-1, 1]$, and

$$p(x) \geqslant \frac{1}{(1 - x^2)^{1/2}}\ ? \tag{13.3}$$

It was noticed by Feldheim in 1938 that, for an appropriate f,

$$\int_{-1}^{-1} [f(x) - L_n(f, P(\tfrac{1}{2}, \tfrac{1}{2}))]^4\, dx$$

is unbounded. The general case of $\int_{-1}^{-1} [f(x) - L_n(f, P(\alpha, \beta))]^{2k}\, dx$ was treated by Askey in [1].

14. Relation (12.3) can be written as

$$\lim_{n\to\infty} \int_{-1}^{1} (L_n(f, A) - f(x))\, dx = 0 \tag{14.1}$$

for all $f \in C[-1, 1]$. Now we raise the question: what are necessary and sufficient conditions for A in order that

$$\lim_{n\to\infty} \int_{-1}^{1} |L_n(f, A) - f(x)|^{\lambda}\, dx = 0 \tag{14.2}$$

for every $f(x) \in C[-1, +1]$. For $\lambda = 1$, one can show that (in the notation of (12.1)) in addition to (12.4)

$$\sum_{x_{\nu n} \in I} |\lambda_{\nu n}| < \epsilon \tag{14.3}$$

must also be satisfied for every set I consisting of a finite number of disjoint intervals with total length $\leqslant \delta$,

$$\delta = \delta(\epsilon). \tag{14.4}$$

For $\lambda = 2$, a trivial sufficient condition is

$$\sum_{\nu=1}^{n} \int_{-1}^{1} (l_{\nu n}(x; A))^2\, dx + \sum_{1 \leqslant \nu < \mu \leqslant n} \left| \int_{-1}^{1} l_{\mu n}(x, A)\, l_{\nu n}(x, A)\, dx \right| = O(1), \tag{14.5}$$

and a necessary condition (according to my paper [9] with Erdös) is

$$\sum_{\nu=1}^{n} \int_{-1}^{1} l_{\nu n}(x, A)^2\, dx = O(1). \tag{14.6}$$

So we have

Problem XI. *Given $\lambda > 1$, what is a necessary and sufficient condition that*

$$\lim_{n\to\infty} \int_{-1}^{1} |f - L_n(f, A)|^{\lambda}\, dx = 0$$

for every $f(x) \in C[-1, +1]$?

There are further interesting questions concerning various classes of functions, but it shall not go into details.

15. The next problem requires some more preparation. We mentioned twice above antedecents of the Banach–Steinhaus theorem in approximation theory. A common feeling prevailed that all convergence theorems of interpolation theory are related to the order of magnitude of $\sum_{\nu=1}^{n} |l_{\nu n}(x, A)|$. In the paper [13] with Erdös, we investigated the correctness of this conjecture. It became clear that this conjecture is false if one goes a little beyond continuity.

More specifically, we asked what consequences can be drawn from

$$\overline{\lim_{n\to\infty}} \frac{M_n(A)}{n^{\beta-\epsilon}} < \infty \tag{15.1}$$

and

$$\underline{\lim_{n\to\infty}} \frac{M_n(A)}{n^{\beta-\epsilon}} > 0 \tag{15.2}$$

$(0 < \beta < 1)$ about the behavior of $L_n(f, A)$, if

$$f(x) \in \mathrm{Lip}_\alpha[-1, 1]. \tag{15.3}$$

(The Lipschitz class $\mathrm{Lip}_\alpha[-1, +1]$ consists of the functions f for which $|f(x_2) - f(x_1)| \leqslant K_f \mid x_2 - x_1 \mid^\alpha$, if $-1 \leqslant x_1 \leqslant x_2 \leqslant +1$.) If was not difficult to show that, for

$$0 < \alpha < \frac{\beta}{\beta - 2}, \tag{15.4}$$

there exists an $f_0(x) \in \mathrm{Lip}_\alpha[-1, +1]$ such that

$$\overline{\lim_{n\to\infty}} \max_{-1\leqslant x\leqslant 1} \mid L_n(f_0 ; A) = \infty.$$

In this case we say that the matrix A is "bad" for the Lipschitz class Lip_α. On the other hand it is trivial that, if

$$\beta < \alpha < 1, \tag{15.5}$$

then $L_n(f; A) \to f(x)$ for every $f \in \mathrm{Lip}_\alpha$. We say that the matrix A is "good" for the Lipschitz classes Lip_α satisfying (15.5).

For the Lipschitz classes Lip_α where

$$\frac{\beta}{\beta + 2} < \alpha < \beta, \tag{15.6}$$

everything is possible. In this case there are "good" matrices as well as "bad" ones. Hence, if (15.6) is satisfied, then the behavior of the "Lebesgue constants" $M_n(A)$ does not determine the convergence of Lagrange interpolation polynomials for the Lipschitz classes Lip_α. Such cases where the Lebesgue constants do not determine the convergence behavior of Lagrange interpolation, will be said to belong to the "fine" theory of interpolation. Now an analogous question can be raised for any sequence of linear operators. I confine myself to the theory of mechanical quadrature.

PROBLEM XII. *Let* $0 < \beta < 1$ *be given. Consider the matrices* A *for which*

$$\overline{\lim_{n\to\infty}} \frac{D_n(A)}{n^{\beta+\epsilon}} < \epsilon, \tag{15.7}$$

$$\overline{\lim_{n\to\infty}} \frac{D_n(A)}{n^{\beta-\epsilon}} > 0, \tag{15.8}$$

for every small $\epsilon > 0$. *Find the largest interval*

$$\psi_1(\beta) < \alpha < \psi_2(\beta) \tag{15.9}$$

for which the theory of mechanical quadrature is "fine", that is, for which the class of matrices A *satisfying* (15.7)–(15.8) *contains "good" matrices as well as "bad" ones.*

The existence of such an interval was shown by Szabados [48].

16. Difficulties of a new type arise if we want to extend our theorem (12.9) on mean convergence to an infinite interval. I ran into this problem with J. Balázs in 1961, in connection with a physical problem. The mathematical problem was as follows. What can be said about the Fourier transform of a continuous function $f(x)$, defined for $x \geqslant 0$, whose values have been observed at merely a finite number of points. Since in physics it is common to assume exponential decrease, one can formulate our question as follows: Find an approximation to

$$F(x) \stackrel{\text{def}}{=} \int_0^\infty \varphi(t)\, e^{-t} \cos xt\, dt, \tag{16.1}$$

for $x > 0$, if $\varphi(t)$ is continuous and if its values are known at given points $0 < t_1 < \cdots < t_n$.

We require that the approximating function $F_n(x)$ satisfy the following assumptions:

(a) If $\varphi(t)$ is a polynomial of degree k in t, then, for $n > k$, we have

$$F_n(x) = F(x); \tag{16.2}$$

(b) For $n \to \infty$, we have

$$F_n(x) \to F(x), \tag{16.3}$$

uniformly for $x \geqslant 0$.

We solved this problem taking as points of observation $t_1, t_2, \ldots, t_n$ the zeros of the Laguerre polynomial $L_n^*(t)$ (which are known to be positive and simple) and replaced $\varphi(t)$ by its Lagrange interpolation polynomial belonging to these points. (We use the notation of §4 for the interval $(0, \infty)$). The first requirement is obviously satisfied. Denote by L^* the matrix of the zeros of $L_n^*(x)$, $n = 1, 2, \ldots$. Then (b) would follow if we could prove that

$$\lim_{n\to\infty} \int_0^\infty (\varphi(t) - L_n(\varphi, L^*))^2 e^{-t} dt = 0 \tag{16.4}$$

under the natural assumption that

$$\lim_{t\to\infty} \varphi(t) e^{-at} = 0 \tag{16.5}$$

for some $0 < a < \frac{1}{2}$.

The first mean convergence theorem for a general class of weight functions is contained in my paper [3] with Balázs. If, besides (16.5) we assume only the continuity of φ, we cannot expect more than (16.3). Therefore the following problem arises:

PROBLEM XIII. *If* (16.5) *is satisfied and the modulus of continuity of* $\varphi(t)$ *is given, what can be said about the behavior of* $|F_n(x) - F(x)|$?

17. From the above-mentioned paper with Balázs it becomes clear that one can take

$$F_n(x) = \sum_{\nu=1}^{n} \varphi(x_{\nu n}) \psi_{\nu n}(x), \tag{17.1}$$

where

$$\psi_{\nu n}(x) \overset{\text{def}}{=} \int_0^\infty l_{\nu n}(t, L^*) e^{-t} \cos tx \, dt. \tag{17.2}$$

It is easy to see that this is a rational function in x which can be written explicitly. On the other hand, it is difficult to calculate it for large values of n.

PROBLEM XIV. *Find an asymptotic formula for* $\psi_{\nu n}(x)$, *for* $n \to \infty$, *which holds uniformly in* ν *and in* x.

I think, what is most esssential here is that $F_n(x)$ gives the exact value of $F(x)$ for a "dense" set of $\varphi(t)$. I intend to return later to problems related to this one.

We do not state here separately similar problems for other transforms, for instance, the Hankel transform.

II. Hermite–Fejér Interpolation

18. Various types of questions can be raised, in connection with the inspired remark of Fejér's that, sometimes, conclusions on the matrix A

can be drawn from properties of the fundamental functions, or of the Cotes numbers of various interpolation formulas. For instance, using the fact that the Cotes numbers belonging to the zeros of the Legendre polynomials are non-negative, Fejér obtained the result that the difference between two consecutive zeros of the Legendre polynomial tend to zeros uniformly as $n \to \infty$. In fact, a much stronger statement can be made. I showed this with Erdös in 1938 and 1940 in our papers [10, 11]. For instance, the difference between two consecutive zeros of the Legendre polynomial $p_n^{(0,0)}(x)$ is of the exact order $1/n$. The question whether the assumption (using the notation (12.2))

$$\lambda_{\nu n} \geqslant 0 \qquad (\nu = 1, 2, \dots, n;\ n = 1, 2, \dots) \tag{18.1}$$

gives a non-trivial interval for the zeros, seems to be much more difficult. So we pose

PROBLEM XV. *Suppoxe that* (18.1) *holds. For each pair* (ν, n), *determine the exact interval to which* $x_{\nu n}$ *belongs.*

19. After the discovery of Faber, the following question naturally arose: Does there exist a procedure different from Lagrange's interpolation which is "efficient" for the class $C[-1, 1]$? Immediately after Faber's proof of his theorem, Fejér discovered that the situation changes if we consider Hermite interpolation, that is, the polynomials

$$H_n(x; f; A) = H_n(f, A)$$

of degree at most $2n - 1$, characterized by the properties

$$\begin{gathered} H_n(x_{\nu n}; f; A) = f(x_{\nu n}) \qquad (\nu = 1, 2, \dots, n), \\ \left. \frac{dH_n(x_n; f; A)}{dx} \right|_{x = x_{\nu n}} = y'_{\nu n} \qquad \text{(given)}. \end{gathered} \tag{19.1}$$

These polynomials can be written as

$$H_n(f; A) = \sum_{\nu=1}^{n} f(x_{\nu n})\, h_{\nu n}(x; A) + \sum_{\nu=1}^{n} y'_{\nu n}\, g_{\nu n}(x; A). \tag{19.2}$$

For the fundamental functions of the first and the second kind Fejér found the relations (using the notation of (4.3)–(4.4)):

$$\begin{gathered} h_{\nu n}(x, A) = \left\{ 1 - \frac{\omega_n''(x_{\nu n})}{\omega_n'(x_{\nu n})} (x - x_{\nu n}) \right\} \cdot l_{\nu n}(x, A)^2, \\ g_{\nu, n}(x; A) = (x - x_{\nu n})\, l_{\nu n}(x; A)^2. \end{gathered} \tag{19.3}$$

In 1916 he showed [18] that

$$H_n(f; T) \to f(x), \tag{19.4}$$

uniformly in $[-1, 1]$, for every $f \in C[-1, 1]$, provided that

$$y'_{\nu n} = 0. \tag{19.5}$$

This result was further improved by him in 1930 [20]. Namely, he replaced condition (19.5) by the weaker one

$$y'_{\nu n} = o\left(\frac{n}{\log n}\right), \tag{19.6}$$

uniformly in ν. (This result cannot be improved.) In 1932 Szegö [54] proved a similar result for Jacobi matrices $P(\alpha, \beta)$ (see (5.6)) on $[-1 + \epsilon, 1 - \epsilon]$, assuming

$$y'_{\nu n} = O(1). \tag{19.7}$$

Because of our theorem (12.5), one could expect that there is a general convergence theorem for the matrix $A = P$ corresponding to the weight function $p(x)$ when

$$p(x) \geqslant c > 0. \tag{19.8}$$

Strangely enough, nothing really interesting is known in this direction. In 1954 I noted that there is a convergence theorem if $p(\cos\theta) \sin\theta$ is positive and continuous in $0 \leqslant \theta \leqslant \pi$ and if

$$| f(x_1) - f(x_2) | = O(1) \, | \log | x_1 - x_2 | \, |^{-1-\epsilon}.$$

The reason for this is that the above condition on p assures the validity of the asymptotic formula of S. Bernstein for the orthogonal polynomials $q_n(x)$ belonging to $p(x)$. This result was improved in 1954 by Freud [24] who showed that it is enough to assume that (8.1) is satisfied in a subinterval (a, b) of $[-1, 1]$. Of course, the convergence can be assured only in this interval. The proof is much more difficult. Hence we pose

PROBLEM XVI. *Find a large class of weight functions $p(x)$ for which* (19.5) *implies*

$$H_n(f; P) \to f(x), \tag{19.9}$$

uniformly in $[-1 + \epsilon, 1 - \epsilon]$, *for every* $f \in C[-1, 1]$. *Is* (19.8) *sufficient for* (19.9)?

PROBLEM XVII. *Is condition* (19.8) *sufficient to assure*

$$\lim_{n\to\infty} \int_{-1}^{1} (f - H_n(f, P))^2 \, dx = 0? \tag{19.10}$$

20. One could expect that if, for some matrix A, the corresponding Hermite–Fejér step parabolas $H_n^*(f, A)$ (see (19.2), (19.3), (19.5)) satisfy

$$H_n^*(f, A) \to f(x) \tag{20.1}$$

in $[-1 + \epsilon, 1 - \epsilon]$, for every $f \in C[-1, 1]$, then the nodes of A must be "very regularly" distributed in $[-1, 1]$. I have alluded to such a theorem in §18. An older theorem of a similar character was obtained in the investigation of the following question. Let l be a given closed Jordan curve in the complex plane, and let the elements of A belong to l. Suppose that f is a regular function in the closed interior of l. What is a condition on A which ensures that

$$L_n(f, A) \to f(z) \tag{20.2}$$

uniformly, in every closed subdomain of the interior? Fejér [19] and Kalmár [33] showed that necessary and sufficient conditions are the following: Let

$$w = \emptyset(z) \tag{20.3}$$

map the outside of l one-to-one and conformally onto $|w| > 1$ ($\emptyset$ is continuous on the closed exterior of l). To the elements in the nth row of A there correspond points on $|w| = 1$. The theorems of Fejér and Kalmár asserts that a necessary and sufficient condition for (20.2) is that these n points be uniformly distributed on $|w| = 1$ (in Weyl's sense). We say that $w_{1n}, w_{2n}, \ldots, w_{nn}$ are uniformly distributed on $|w| = 1$ in Weyl's sense, if the number of $w_{\nu n}$ which are on a given arc of the circle $|w| = 1$ divided by n tends to $1/2\pi$ of its length, as $n \to \infty$.

In particular, if l is the interval $[-1, 1]$, then

$$\emptyset^{-1}(w) = \frac{1}{2}\left(w + \frac{1}{w}\right). \tag{20.4}$$

Let the elements of A be denoted by $x_{\nu n}$, and let the image points on $|w| = 1$ be

$$e^{\pm i\theta_{\nu n}} \qquad (0 \leqslant \theta_{\nu n} \leqslant \pi), \tag{20.5}$$

that is, let

$$x_{\nu n} = \cos \theta_{\nu n}. \tag{20.6}$$

Then the points $\theta_{\nu n}$ have to be uniformly distributed in $[0, \pi]$. This can be interpreted geometrically as follows: Let the points $x_{\nu n}$ be projected on the semicircle over $[-1, 1]$. These projections have to be uniformly distributed on the semicircle in order that (20.2) be true for every $f(x)$ analytic in $[-1, +1]$.

Now we consider the following question. For a given $0 < \alpha < 1$, what is a necessary condition on the nth row of A in order that

$$L_n(f; A) \to f(x) \tag{20.7}$$

on $[-1, 1]$ for every

$$f \in \mathrm{Lip}_\alpha[-1, +1]. \tag{20.8}$$

It follows from (15.4) and (15.1) that for $\epsilon > 0$ the inequalities

$$\sum_{\nu=1}^{n} |l_{\nu n}(x; A)| < c(\epsilon)\, n^{(2\alpha-1)/(1-\alpha)-\epsilon}$$

and

$$\sum_{\nu=1}^{n} |l_{\nu n}(x; A)| < cn^{2\alpha/(1-\alpha)}$$

must hold. This implies that, with some constants c, $\gamma = \gamma(\alpha)$, we have

$$|l_{\nu n}(x; A)| < cn^{\gamma}, \qquad \nu = 1, 2,..., n, \quad n = 1, 2,.... \tag{20.9}$$

Hence, according to a theorem in my paper with Erdös [11, Theorem XV], we have

$$\left| \sum_{a \leqslant \theta_{\nu n} \leqslant b} 1 - \frac{a-b}{\pi} n \right| < c(\alpha, \epsilon)\, n^{1/2+\epsilon}; \tag{20.10}$$

that is, (20.7) and (20.8) imply that the $\theta_{\nu n}$ are uniformly distributed in intervals of length $n^{2\epsilon-1/2}$.

In view of the next problem, Theorem XIV of our above-mentioned paper is even more surprising. According to this theorem, from

$$|l_{\nu n}(x, A)| \leqslant T = \text{Const}, \qquad \nu = 1, 2,..., n, \quad n = 1, 2,..., \tag{20.11}$$

it follows that

$$\sum_{a \leqslant \theta_{\nu n} \leqslant b} \left| 1 - \frac{a-b}{\pi} n \right| \leqslant c(T, \epsilon)((a-b)\, n)^{1/2+\epsilon}. \tag{20.12}$$

This means that the elements of a row in A are uniformly distributed in intervals of length $\Omega(n)/n$ where $\Omega(n)$ is a function tending to ∞ arbitrarily slowly.

If, for the Hermite–Fejér interpolation polynomials $H_n^*(f; A)$, we have (20.1) uniformly in $[-1, +1]$, then it is obviously necessary that

$$\max_{1 \leqslant \nu \leqslant n} \max_{-1 \leqslant x \leqslant 1} | h_{\nu,n}(x; A)| \leqslant T, \qquad n = 1, 2,.... \tag{20.13}$$

Now we ask

PROBLEM XVIII. *What kind of uniform distribution does the restriction* (20.13) *imply for the* $\theta_{\nu n}$ *defined by* (20.6)?

Certainly it is at least as strong as (20.12).

21. After the discussion of §15, we can at once state

PROBLEM XIX. *Do the Hermite–Fejér "step parabolas" have a "fine" convergence theory*?

It is worthwhile to state a second part of this problem as a separate one.

PROBLEM XX. *Suppose that A is a matrix satisfying*

$$\sum_{\nu=1}^{n} | h_{\nu n}(x, A)| \leqslant Cn^{\beta}(0 < \beta < 1), \tag{21.1}$$

where C is independent of n. What is the greatest lower bound of the set of α's for which (20.1) *holds for all $f(x)$ satisfying*

$$f(x) \in \mathrm{Lip}_{\alpha}[-1, +1]?$$

Section 16 makes the following problem interesting.

PROBLEM XXI. *Let $H^*(f, L^*)$ be the nth Hermite–Fejér interpolation polynomial of $f(x)$ based on the Laguerre matrix L^*. Is it true that*

$$\lim_{n\to\infty} \int_0^{\infty} (f(t) - H_n^*(f, L^*))^2 \, e^{-t} \, dt = 0$$

for every continuous f satisfying (16.5)? *Theorem* 14.7 *of Szegö's monograph* [56] "*Orthogonal Polynomials*" *suggests that the answer to this question is affirmative.*

22. The results of Fejér and Szegö on convergence of the Hermite–Fejér stepparabolas convey the impression that the convergence behavior

of Hermite–Fejér interpolation is always better than that of Lagrange interpolation. Our next problem is in this direction.

PROBLEM XXII. *Let $0 < \alpha < 1$ be given. Find a matrix A^* such that, for all $f \in \mathrm{Lip}_\alpha(-1, 1)$, we have*

$$L_n(f, A^*) \to f, \tag{22.1}$$

uniformly in $[-1, -1]$, whereas for some f^* we have

$$\overline{\lim_{n\to\infty}} \max_{-1\leqslant x\leqslant 1} H^*(f^*, A^*)| = \infty.$$

If such a matrix exists, it would mean that Lagrange interpolation may be "good" for a large class of functions for which Hermite–Fejér interpolation is not a good means of approximation. On the other hand, can it happen that the step parabolas belonging to a given matrix A are "much worse" than the Lagrange parabolas belonging to the same A? Thus we are led, for instance, to the following

PROBLEM XXIII. *Let $0 < \alpha < 1$ be given. Suppose that A^* is such that $L_n(f, A^*) \to f$ uniformly in $[-1, 1]$ for every $f \in \mathrm{Lip}_\alpha[-1, 1]$. Is it true that there exists an integer r such that if g is r-times continuously differentiable in $[-1, 1]$, then $H_n^*(g, A^*) \to g$ uniformly in $[-1 - \epsilon, 1 - \epsilon]$?*

An affirmative answer to this question seems to be the case because of the fact that from our assumption follows, as in §20, that

$$\sum_{\nu=1}^{n} l_{\nu,n}(x, A^*)| < cn^{2\alpha\cdot(1-\alpha)}, \qquad -1 \leqslant x \leqslant 1. \tag{22.2}$$

23. The first theorem drawing a general conclusion from the behavior of the polynomials $H^*(f, A)$ on those of $L(f, A^*)$ was found by Fejér. He calls a matrix A "strongly normal" if, for all n and for $\nu = 1, 2,..., n$, we have

$$1 - \frac{\omega_n''(x_{\nu,n})}{\omega_n'(x_{\nu,n})}(x - x_{\nu,n}) \geqslant \rho > 0, \qquad -1 \leqslant x \leqslant 1, \tag{23.1}$$

where ρ is independent of ν and n. He proved that in this case,

$$L_n(f, A) \to f,$$

uniformly in $[-1, -1]$, if $f \in \mathrm{Lip}_\alpha[-1, +1]$, $\alpha > \frac{1}{2}$. In his posthumous paper [31], Grünwald showed that, for such a matrix A,

$$H_n^*(f, A) \to f, \tag{23.2}$$

uniformly in $[-1, +1]$, for all $f \in C[-1, +1]$.

It is likely that (23.2) alone can assure that $L_n(f, A)$ cannot behave "too badly." Such a conjecture can be formulated as

PROBLEM XXIV. *Is it true that, for any matrix* A^* *satisfying* (23.2), *we have* $L_n(f, A^*) \to f$ *for all functions* f *which are continuously differentiable in* $[-1, 1]$?

24. It is natural to ask about the "real reason" for theorem (19.4)–(19.5) of Fejér. Thinking geometrically one could imagine that by letting the derivative be zero, we prevent the interpolation polynomials from "jumping". If it were so, then by *not* prescribing a value of the derivative at a single point of A, we would not change the situation too much. Of course, the degree of the interpolation polynomial

$$H_n^{**}(f, T) \tag{24.1}$$

would then be $\leqslant 2n - 2$. Call the point $x_{\nu(n)n} = x_{\nu(n)}$ for which the value of the derivative is not prescribed, the exceptional point. At the end of the thirties I raised the question to my friend E. Feldheim, How do the interpolation polynomials behave in $[-1, +1]$ if $\lim_{n\to\infty} x_{\nu(n)n} \to \xi$, ξ being an interior point of $[-1, 1]$? Feldheim found that the polynomials converge uniformly in the two intervals we get by removing an arbitrary small neighborhood of ξ from $[-1, +1]$.

In my paper [60] dedicated to the memory of Fejér, I described a peculiar situation concerning the critical point. The polynomials $H_n^{**}(f_0, T)$ are uniformly bounded in $[-1, +1]$, but for some $f_0(x) \in C[-1, +1]$ and $\xi = \cos(\pi/5)$ they do not converge.

One can ask

PROBLEM XXV. *Can one distribute the exceptional points in* $[-1, +1]$ *so that, with some* $f_1(x) \in C[-1, +1]$, *the polynomials* (24.1) *would be uniformly bounded in* $[-1, 1]$ *and would diverge everywhere*?

Since problems concerning further peculiarities of $H^{**}(f, T)$ have been solved by Vértesi [63] and by Meir *et al.* [38], I end my discussion here.

25. I return to the theorem of Gauss already mentioned in §1 which states that if $A = P^*$ is the matrix of the zeros $x_{\nu n}^*$ of the Legendre polynomials (or using the notations of (5.6), if $P^* = P(0, 0)$), then the relation

$$\int_{-1}^{1} \pi(x)\, dx = \sum_{\nu=1}^{n} \pi(x_{\nu n}^*) \int_{-1}^{1} l_{\nu n}(x; P)\, dx \tag{25.1}$$

is true not only for polynomials $\pi(x)$ of degree $\leqslant n - 1$ but even for polynomials of degree $\leqslant 2n - 1$. As a further preparation to our next subject I mention Hermite interpolation, according to which, if $m_1, \ldots, m_n$ are arbitrary natural numbers, then for the points (4.1) there is exactly one polynomial $G(x)$ of degree not exceeding

$$m_1 + m_2 + \cdots + m_n - 1 \overset{\text{def}}{=} N. \tag{25.2}$$

for which

$$G^{(j)}(x_{\nu n}) = a_{\nu nj} \qquad (j = 0, 1, \ldots, m_\nu - 1, \nu = 1, \ldots, n). \tag{25.3}$$

If

$$m_1 = m_2 = \cdots = m_n = m \tag{25.4}$$

(this is the case of main interest to us), then $G(x)$ can be written as

$$\begin{aligned} G(x) &= \sum_{\nu=1}^{n} G(x_{\nu n})\, l_{\nu n0}(x; A) \\ &+ \sum_{\nu=1}^{n} G'(x_{\nu n})\, l_{\nu n1}(x; A) + \cdots + \sum_{\nu=1}^{n} G^{(m-1)}(x_{\nu n})\, l_{\nu n, m-1}(x; A). \end{aligned} \tag{25.5}$$

where $l_{\nu nj}(x; A)$ are the fundamental functions of Hermite interpolation. Hence, the formula

$$\begin{aligned} \int_{-1}^{-1} G(x)\, dx &= \sum_{\nu=1}^{n} G(x_{\nu n}) \int_{-1}^{1} l_{\nu n0}(x; A)\, dx + \cdots \\ &+ \sum_{\nu=1}^{n} G^{(m-1)}(x_{\nu n}) \int_{-1}^{+1} l_{\nu n, m-1}(x; A)\, dx \end{aligned} \tag{25.6}$$

is exact if $G(x)$ is a polynomial of degree at most $mn - 1$ and the points (4.1) are arbitrary. The numbers

$$\int_{-1}^{1} l_{\nu nj}(x, A)\, dx \overset{\text{def}}{=} \lambda_{n\nu j} \qquad (j = 0 \cdots m - 1;\ \nu = 1, \ldots, n, n = 1, 2, \ldots) \tag{25.7}$$

will be called Cotes numbers of higher order.

26. Because of the theorem of Gauss it is natural to ask whether knots (4.1) can be chosen so that the quadrature formula (25.6) will be exact for polynomials of degree not exceeding $(m + 1)\, n - 1$. In my paper [59], which appeared in 1950, I showed that the answer is negative for $m = 2$ positive, and it is for $m = 3$. Furthermore, I proved that the uniquely

determined matrix A consists of the zeros of the polynomial $\pi_n^*(x)$ which minimizes the integral

$$\int_{-1}^{1} \pi_n(x)^4 \, dx, \tag{26.1}$$

where

$$\pi_n(x) = x^n + \cdots. \tag{26.2}$$

More generally, the answer is negative for even, and positive for odd m. The unique matrix A, for odd m, is given by the zeros of the polynomials minimizing

$$\int_{-1}^{1} \pi_n(x)^{m-1} \, dx. \tag{26.3}$$

It is known and also directly provable that these zeros are all simple and contained in the interval $(-1, +1)$. Gauss's theorem follows, for $m = 1$, by a known extremum property of the Legendre polynomials.

Little is known about the extremal polynomials of (26.3) for $m \geqslant 3$. I shall return to this question. Instead of (25.6), it is also interesting to investigate the analogous formula

$$\int_{-1}^{1} G(x)\, p(x)\, dx = \sum_{\nu=1}^{n} G(x_{\nu n}) \int_{-1}^{1} l_{\nu n 0}(x, A)\, p(x)\, dx + \cdots, \tag{26.4}$$

with a weight function $p(x)$ as in §5. Then the role of the integral (26.3) is taken over by

$$\int_{-1}^{1} \pi_n(x)^{m+1}\, p(x)\, dx. \tag{26.5}$$

Particularly interesting is the case

$$p(x) = (1 - x^2)^{-1/2}. \tag{26.6}$$

By a theorem of S. Bernstein, in this case, the nth Chebyshev polynomial is the minimizing polynomial for odd values of m. The formula

$$\int_{-1}^{1} \frac{G(x)}{(1 - x^2)^{1/2}}\, dx = \sum_{\nu=1}^{n} G\left(\cos \frac{2\nu - 1}{2n}\pi\right) \int_{-1}^{1} \frac{l_{\nu n 0}(x; T)}{(1 - x^2)^{1/2}}\, dx + \cdots \tag{26.7}$$

is exact for polynomials $G(x)$ of degree not exceeding $(m + 1)\, n - 1$. Since, as I remember, formula (26.7) is used in methods of Runge–Kutta type, the following problem seems to be interesting.

PROBLEM XXVI. *Give an explicit formula for*

$$\int_{-1}^{1} \frac{l_{\nu nj}(x, T)}{(1 - x^2)^{1/2}}\, dx$$

and determine its asymptotic behavior as $n \to \infty$.

27. Before proving his convergence theorem, mentioned in §19, Fejér investigated the step parabolas in the classical case where the knots are the zeros of Legendre polynomials. He found that the convergence is uniform in $[-1 + \epsilon, 1 - \epsilon]$, and at $x = +1$ and $x = -1$ the step parabolas tend to $\frac{1}{2} \int_{-1}^{1} f(x)\, dx$. In my joint paper [7] with Egerváry, we observed that if the step parabolas are replaced by the polynomials of degree $\leqslant 2n - 1$ taking the values of the function and of its derivative at the Legendre zeros, and the values of the function at $x = \pm 1$, then the convergence becomes uniform. This theorem was generalized by Szász [53] in 1959 and by Berman [6] in 1973. For arbitrary Jacobi matrices $P(\alpha, \beta)$, the question is not yet settled. For general weight functions, nothing is known. Therefore we can ask the following two questions.

PROBLEM XXVII. *Find a class of weight functions* $p(x)$ *such that, for the matrix* P *arising from* $p(x)$, *and for the polynomials* $E_{2n+1}(f, P)$ *given by*

$$E_{2n+1}(f, P) = f(x_{\nu n}),$$
$$\frac{dE_{2n+1}(f, P)}{dx} = 0 \qquad (x = x_{\nu n}),$$
$$E_{2n+1}(f, P)|_{x=\pm 1} = f(\pm 1),$$

the limit relation

$$\lim_{n\to\infty} E_{2n+1}(f, P) = f(x)$$

holds uniformly in $[-1, +1]$.

PROBLEM XXVIII. *Give a general class of matrices* P *such that, for all* $f \in C[-1, +1]$,

$$\lim_{n\to\infty} \int_{-1}^{1} (f - E_{2n+1}(f, P))^2\, p(x)\, dx = 0.$$

III. BIRKHOFF OR LACUNARY INTERPOLATION

28. The basic problem of Hermite interpolation is the determination of the polynomial $\pi(x)$ of minimal degree for which

$$\pi^{(k)}(x_j) \text{ can be prescribed } (k = 0, 1,..., m_j - 1, j = 1, 2,..., n). \qquad (28.1)$$

That is, the consecutive derivatives are prescribed. G.D. Birkhoff, in 1906, was the first to consider the general case, where we drop the requirement of being consecutive. While polynomials of the previous kind always exist, in Birkhoff's case, polynomials satisfying his conditions do not necessarily exist. Hence, we have the basic questions:

$$\begin{array}{ll} \text{(a)} & \text{existence,} \\ \text{(b)} & \text{uniqueness,} \\ \text{(c)} & \text{possibly, explicit representation,} \\ \text{(d)} & \text{convergence,} \\ \text{(e)} & \text{applications.} \end{array} \tag{28.2}$$

Birkhoff assumed (a) and (b) and was mostly interested in (e), for instance in studying the error term in mechanical quadrature. In the middle of the 1930's, I had a conversation with Fejér on interpolation. I mentioned to him that it would be interesting to investigate, for the matrix T, the sequence of polynomials of degrees not exceeding $2n - 1$, for which the values of the function and those of the second derivative are given at the knots. (One calls this (0, 2) interpolation.) The only work in this direction he knew was a paper of Pólya of 1931. He did not know of Birkhoff's work. Having looked at Birkhoff's paper, I realized that he did not consider questions of convergence. I postponed study of this question to complete my current investigations. Then events of world history intervened so that I was able to carry out this study only in 1955. Since we did not have any matrix for which existence and uniqueness of (0, 2) interpolation polynomials were known, I analyzed with Surányi [47] the case where the knots are the zeros of the ultraspheric polynomials $P_n^{(\alpha,\alpha)}(x)$, including the case $\alpha = -1$. It turned out that there can be uniqueness only for

$$n = 2k, \tag{28.3}$$

but even in this case, it is not always guaranteed.

This motivates the following

PROBLEM XXIX. *Find all Jacobi matrices $P(\alpha, \beta)$, $\alpha \neq \beta$, for which the* (0, 2) *interpolation problem does have a unique solution.*

(29. If in the nth row of a matrix A there are n interpolation points, then A is called "very good" if, for arbitrary sets of numbers $y_{\nu n}$ and $y''_{\nu n}$, there is a uniquely determined polynomial $D_n(f; A) = D_n(f)$ of degree at most $2n - 1$ for which

$$\begin{gathered} D_n(f; A)_{x=x_{\nu n}} = y_{\nu n} = f(x_{\nu n}), \\ \left(\frac{d^2 D_n(f; A)}{dx^2}\right)_{x=x_{\nu n}} = y''_{\nu n}\,. \end{gathered} \tag{29.1}$$

In that case, $D_n(f; A)$ can be uniquely written as

$$D_n(f, A) = \sum_{\nu=1}^{n} f(x_{\nu n})\, r_{\nu n}(x; A) - \sum_{\nu=1}^{n} y''_{\nu n} \rho_{\nu n}(x; A), \tag{29.2}$$

where

$$\begin{aligned} r_{\nu n}(x_{jn}) &= 1 && \text{if } j = \nu, \\ &= 0 && \text{otherwise;} \\ r''_{\nu n}(x_{jn}) &= 0 && (j = 1, 2,\dots, n); \end{aligned} \tag{29.3}$$

and

$$\begin{aligned} \rho_{\nu n}(x_{jn}) &= 0 && (j = 1, 2,\dots, n); \\ \rho''_{\nu n}(x_{j,n}) &= 1 && \text{if } \nu = j, \\ &= 0 && \text{otherwise.} \end{aligned} \tag{29.4}$$

The polynomials $r_{\nu,n}(x)$ and $\rho_{\nu,n}(x)$ are called the fundamental functions of the first and second kind of the interpolation procedure.

30. It turns out that not the T-matrix but rather the π-matrix is the "handiest" for the problem, even when the restriction (28.3) is needed. The kth row of this matrix π is given by the zeros of the polynomial

$$\pi_{2k}(x) = \int_{-1}^{x} P_{2k-1}(t)\, dt \tag{30.1}$$

$P_{2k-1}(t)$ being the $(2k - 1)$-th Legendre polynomial; in particular $x_{1,2k} = 1$, $x_{2k.2k} = -1$.

I published the first theorem on convergence with Balázs in 1958 [2]. I shall not go into details on this subject. I want only to mention that there is some freedom in choosing the $y''_{\nu n}$. Namely, we need only the restriction

$$y''_{\nu n} = o(n) \qquad \text{as } n \to \infty. \tag{30.2}$$

This restriction cannot be weakened.

31. Before proceeding, I would like to make some general remarks on the theory of lacunary interpolation.

In his report "Birkhoff Interpolation Problem" (Center for Numerical Analysis, The University of Texas at Austin, 1975), G.G. Lorentz very nicely summarizes and complements the litarature on the problem. He is mostly interested in questions of regulairty, namely, characterizing those natural numbers

$$0 \leqslant k_{1j} < k_{2j} < \cdots < k_{l_j j} \qquad (j = 1, 2,\dots, n) \tag{31.1}$$

(n given) for which, with arbitrary choice of the knots

$$(1 \geqslant) \xi_1 > \xi_2 > \cdots > \xi_n (\geqslant 1) \tag{31.2}$$

the polynomial $\pi(x)$ of degree at most $l_1 + l_2 + \cdots + l_n - 1$ is uniquely determined by the relations

$$\pi^{(k_{\nu j})}(\xi_j) = y_{j\nu}, \qquad \nu = 1, 2,..., l_j ; j = 1, 2,..., n, \tag{31.3}$$

for each choice of the $y_{j\nu}$.

This problem is important even if n does not tend to ∞. In fact, the question is interesting even for $n = 2$, a case solved by Pólya. As stated on p. 79 of Lorentz's report, the complete solution of this nice question (originating with Schoenberg) is hopeless. Lorentz also mentions that Birkhoff was not interested in problems of regularity, even though his results contained some sufficient conditions for that. When mentioning the theory of convergence, Lorentz refers to my work with Balázs [2] as the first results. About these and many other related results concerning similar matrices, he says that they all depend upon a very special selection of knots, for which explicit formulas are possible. It is worthwhile to reproduce here the reason for our selection of knots, indicated also in [2].

We look for the global solution of the classical differential equation

$$y''(x) - \varphi(x)\, y(x) = 0 \tag{31.4}$$

on the positive real line. Let

$$0 < \eta_{1n} < \eta_{2n} < \cdots < \eta_{nn}, \tag{31.5}$$

and let A_1 be the matrix belonging to these values. Then, with $y_{\nu n}$'s to be determined later, and with $y''_{\nu n} = \varphi(x_{\nu n})\, y_{\nu n}$, the polynomial

$$D_n(y, A_1) = \sum_{\nu=1}^{n} y_{\nu n}[r_{\nu n}(x; A_1) + \varphi(x_{\nu n})\, \rho_{\nu n}(x; A_1)]$$

satisfies Eq. (31.4) at $\eta_{\nu n}$ for any choice of the $y_{\nu n}$'s. Put

$$D''_n(x) - \varphi(x)\, D_n(x) \overset{\text{def}}{=} \sum_{\nu=1}^{n} y_{\nu n}\, g_{\nu n}(x; A_1). \tag{31.6}$$

Let the initial conditions be, for instance,

$$y(0) = 1, \qquad y'(0) = 0. \tag{31.7}$$

Then there are two linear relations between the $y_{\nu n}$'s. Subject to these relations we have to minimize the quadratic form

$$\int_0^\infty (D_n'' - \varphi D_n)^2 \psi(x)\, dx. \tag{31.8}$$

It can be expected that, for $n \to \infty$, the interpolation polynomials D_n converge to the solution of (31.4) with the initial conditions (31.7). (There are many ways of modification and the initial conditions (31.7) can be replaced by other conditions.) If we want to be able to handle (3.18), we need control over the integrals

$$\int_0^\infty g_{\nu_1 n}(x; A_1)\, g_{\nu_2 n}(x; A_1)\, \psi(x)\, dx. \tag{31.9}$$

We can expect to have this control if we can calculate the functions $g_{\nu n}(x; A_1)$ or the fundamental functions $r_{\nu,n}(x; A_1)$ and $\rho_{\nu n}(x; A_1)$. Therefore, the phrase "very special knots" refers to looking for an explicit basic matrix for which the fundamental functions have a simple form. Such investigations can give valuable information even if they do not give a final answer.

We shall make a further remark in §38.

32. If we take as knots the zeros of $\pi_{2k-1}(x)$ (see (30.1)), then we have a rather unusual case in the theory of convergence of interpolation processes. For odd n, there are infinitely many polynomials with the required properties. So we have

PROBLEM XXX. *Investigate the general theory of lacunary (Birkhoff) interpolation processes for* $f \in C[-1, +1]$.

With his theorems (19.4)–(19.6), Fejér settled the problem of convergence in the "simply infinite" process, where there are bounds only on $|y'_{\nu n}|$.

33. In §28 we defined "very good" matrices. We say now that a matrix A is "good" if, for $\nu = 1, 2, \ldots, n$, and $n = 1, 2, \ldots$, there exists at last one set of functions $r_{\nu n}(x; A)$ and $\rho_{\nu n}(x; A)$ with the properties (29.3) and (29.4). The question of the "most stable" (0, 2)-interpolation is the following

PROBLEM XXXI. *Which "good" matrix A will minimize*

$$\max_{-1 \leqslant x \leqslant 1} \sum_{\nu=1}^{n} |r_{\nu n}(x, A)|\, ? \tag{33.1}$$

Let π_1 denote the matrix defined by (30.1). I showed with Balázs [2] that

$$\max_{-1 \leqslant x \leqslant 1} \sum_{\nu=1}^{n} |r_{\nu n}(x; \pi_1)| \leqslant c_1 n, \tag{33.2}$$

where c_1 is a numerical constant. This cannot be improved since

$$\max_{-1\leqslant x\leqslant 1} \sum_{\nu=1}^{n} | r_{\nu n}(x; \pi_2)| \geqslant c_2 n. \tag{33.3}$$

I believe that for any "good" matrix A,

$$\max_{-1\leqslant x\leqslant 1} \sum | r_{\nu n}(x; A)| > c_3 n. \tag{33.4}$$

If this is true, then the matrix $\pi_1(x)$ is not "far from the optimal A". (A bibliography on Hermite–Birkhoff interpolation was compiled at the end of 1975 by P. L. J. van Rooij, F. Schurer, and van Walt von Praag.)

34. Fejér's theorem mentioned in §19 gives a great freedom in choosing the points $y'_{\nu n}$ without "spoiling" the convergence. That (19.6) is sufficient follows immediately from Fejér's theorem stating that

$$\max_{-1\leqslant x\leqslant -1} \sum_{\nu=1}^{n} | g_{\nu n}(x; T)| = (1 + o(1)) \frac{2}{\pi n} \log n. \tag{34.1}$$

The question which naturally arises is whether this freedom in choosing $y'_{\nu n}$ is the best possible in Fejér's result, that is, whether or not we can allow more than (19.6) for $y'_{\nu n}$. This question, that is, the problem of the "freest" (0, 1)-interpolation, is equivalent o finding a matrix A minimizing

$$\max_{-1\leqslant x\leqslant 1} \sum_{\nu=1}^{n} | g_{\nu n}(x; A)| . \tag{34.2}$$

We answered this question, at least asymptotically, in a paper with Erdös mentioned in §7. We showed that, for any A,

$$\max_{-1\leqslant x\leqslant 1} \sum_{\nu=1}^{n} | g_{\nu n}(x; A)| > \frac{2}{\pi n} (\log n - c \log\log n), \tag{34.3}$$

that is, T gives asymptotically the best result for the "freest" (0, 1) approximation. The corresponding question for (0, 2) interpolation is the following

PROBLEM XXXII. *Which is the "good" matrix A, minimizing*

$$\max_{-1\leqslant x\leqslant -1} \sum | \rho_{\nu n}(x, A)| ?$$

Rahman, Schmeisser, and myself [42] showed that, with some constant c,

$$\max_{-1\leqslant x\leqslant +1} \sum_{\nu=1}^{n} |\rho_{\nu n}(x; A)| > \frac{c}{n^2} \tag{34.4}$$

for every "good" matrix A.

For the matrix π defined in §30,

$$\frac{c_1}{n} \leqslant \max_{-1\leqslant x\leqslant +1} \sum_{\nu=1}^{n} |\rho_{\nu n}(x; \pi)| < \frac{c_2}{n}. \tag{34.5}$$

The plausible and apparently difficult conjecture is that, for any "good" matrix,

$$\max_{-1\leqslant x\leqslant 1} \sum_{\nu=1}^{n} |\rho_{\nu n}(x; A)| > \frac{c_3}{n}, \tag{34.6}$$

which, essentially, cannot be improved.

35. Let A be "very good" in the sense of §28. Then we have, for every polynomial of degree at most $2n-1$,

$$\begin{aligned}\int_{-1}^{+1} \pi_{2n-1}(x)\,dx = \sum_{\nu=1}^{n} \pi_{2n-1}(x_{\nu n}) \int_{-1}^{+1} r_{\nu n}(x; A)\,dx \\ + \sum_{\nu=1}^{n} \pi''_{2n-1}(x_{\nu n}) \int_{-1}^{1} \rho_{\nu n}(x; A)\,dx.\end{aligned} \tag{35.1}$$

The question arises whether we can choose A so that (35.1) remains valid for polynomials of higher degree. This can be formulated as

Problem XXXII. *Determine the matrices A, if any, for which* (35.1) *holds for all polynomials of degree $\leqslant 2n$.*

36. As we have seen in §15, Lagrange interpolation has both a "coarse" theory and a "fine" one of convergence and divergence. For the Hermite–Fejér interpolation polynomials $H_n^*(f; A)$, if we assume (21.1) and

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{\beta-\epsilon}} \max_{x} \sum_{\nu=1}^{n} |h_{\nu n}(x, A)| > 0, \tag{36.1}$$

then we can see that the procedure is "bad" for the class $\mathrm{Lip}_\alpha[-1, +1]$, for all α satisfying

$$0 < \alpha < \frac{\beta}{\beta+2}. \tag{36.2}$$

Thus, these classes fall into the "coarse" theory. A slight generalization of Problem XX makes it plausible that the classes (36.2) give the entire "coarse" theory of Hermite–Fejér interpolation. The analogous questions for (0, 2) interpolation seem to be more complicated because of (33.4). That inequality suggests an affirmative answer to the following

PROBLEM XXXIV. *Is it true that, for every given "very good" matrix A, and for every $\delta > 0$, there exists an*

$$f_0(x) \in \mathrm{Lip}_{1-\delta}[-1, +1] \tag{36.3}$$

for which

$$\overline{\lim_{n\to\infty}} \max_{-1\leq x\leq +1} \sum |f_0(x_{\nu,n})\, r_{\nu n}(x; A)| = \infty\,? \tag{36.4}$$

Even if conjecture (33.4) is true, (36.4) can be proved only for $\frac{2}{3} < \delta < 1$ if we follow the proof of (15.4).

37. If there is an affirmative answer to the previous question, then in the convergence theory of (0, 2)-interpolation, the role of the classes (15.3) is taken by functions $f(x)$ which are continuously differentiable in $[-1, +1]$, and for which

$$f'(x) \in \mathrm{Lip}_\alpha[-1, +1]. \tag{37.1}$$

Here is a problem corresponding to Problem XX.

PROBLEM XXXV. *Suppose that for a "very good" matrix A we have*

$$\max_{-1\leq x\leq +1} \sum_{\nu=1}^{n} |\, r_{\nu n}(x; A)| < n^{\beta}. \tag{37.2}$$

Find the vlaues of α for which $f' \in \mathrm{Lip}_\alpha[-1, +1]$ implies

$$D_n^*(f, A) \overset{\mathrm{def}}{=} \sum_{\nu=1}^{n} f(x_{\nu n})\, r_{\nu n}(x, A) \to f(x), \tag{37.3}$$

uniformly in $[-1, +1]$.

It is likely that, except for the last remark in §33, (0, 2)-interpolation does not have a "coarse" convergence theory.

38. As mentioned in §28, in his paper, Birkhoff obtained for arbitrary "very good" matrices a general formula for the error term in mechanical quadrature. Without mentioning here some disadvantages of his remainder term, we merely note that it involves the $2n$th derivative of the function. On the other hand, my theorem with Balázs gives, in the case of the special

π-matrix, convergence of the quadrature for functions $f(x)$ which are differentaible and whose derivative belongs to a Lipschitz class with arbitrarily small exponent. Connected with this is the following

PROBLEM XXXVI. *What is the "best" class of functions for which the integrals of the polynomials*

$$\sum_{\nu=1}^{n} f(x_{\nu n})\, r_{\nu n}(x, \pi) \qquad (n \text{ even})$$

tend to $\int_{-1}^{1} f(x)\, dx$?

39. The previous discussion could be completed in the negative direction by an affirmative answer to the following

PROBLEM XXXVII. *Does there exist, for every "good" matrix* A, *a function* $f_0(x) \in C[-1, -1]$ *such that, with the notation* (37.3),

$$\overline{\lim_{n\to\infty}} \left| \int_{-1}^{1} D_n^*(f; A)\, dx \right| = \infty? \tag{39.1}$$

Perhaps even the existence of such an $f_0(x; A) \in \mathrm{Lip}_\alpha[-1, +1]$ can be established.

A classical theorem of Steklov [46] and Fejér [21] guarantees that, if a matrix A satisfies

$$\int_{-1}^{-1} l_{\nu n}(x; A)\, dx \geqslant 0, \tag{39.2}$$

then

$$\lim_{n\to\infty} \int_{-1}^{1} L_n(f, A)\, dx = \int_{-1}^{1} f(x)\, dx \tag{39.3}$$

for every Riemann–integrable f. By analogy, one could expect that it is advantageous to study mechanical quadrature for (0, 2) interpolation with matrices A satisfying

$$\int_{-1}^{1} r_{\nu n}(x; A)\, dx \geqslant 0, \qquad \nu = 1, 2, \ldots, n,\ n > n_0\,. \tag{39.4}$$

Our next problem is related to this.

PROBLEM XXXVIII. *Does there exist a matrix* A *satisfying* (39.4)?

In the case of a "very good" matrix A, an affirmative answer to Problem

XXXVII would give a negative answer to Problem XXXVIII. Namely, for such a matrix A,

$$\sum_{\nu=1}^{n} r_{\nu n}(x; A) \equiv 1.$$

Thus, if $|f| \leqslant 1$, then

$$\left| \int_{-1}^{1} D_n^*(f; A)\, dx \right| \leqslant \sum_{\nu=1}^{n} \left| \int_{-1}^{1} r_{\nu n}(x; A)\, dx \right| = \sum_{\nu=1}^{n} \int_{-1}^{1} r_{\nu n}(x; A)\, dx = 2.$$

which contradicts (39.1).

"Good" matrices with the extremal property of the following problem certainly play an exceptional role.

PROBLEM XXXIX. *Determine the "good" matrices for which*

$$\sum_{\nu=1}^{n} \left| \int_{-1}^{+1} r_{\nu n}(x; A)\, dx \right|$$

is minimal.

An affirmative answer to the following question would be very useful.

PROBLEM XL. *Is it true that, for "good" matrices A,*

$$\frac{\max_{-1\leqslant x\leqslant +1} \sum |\rho_{\nu n}(x; A)|}{\max_{-1\leqslant x\leqslant +1} \sum |r_{\nu n}(x; A)|} < \frac{c}{n^2}? \tag{39.5}$$

If it is, then because of the π-matrix, it cannot be essentially improved. A somewhat stronger conjecture is given in

PROBLEM XLI. *Is it true that, for every good matrix A,*

$$\max_{\nu=1,2,\ldots,n} \frac{\max_{-1\leqslant x\leqslant +1} |\rho_{\nu n}(x; A)|}{\max_{-1\leqslant x\leqslant +1} |r_{\nu n}(x; A)|} < \frac{c}{n^2}?$$

IV. INTERPOLATION ON CURVES

40. So far we dealt with interpolation on the interval $[-1, +1]$. Now we study interpolation on a Jordan curve or arc l lying in the complex plane. The theorems of Fejér and Kalmár mentioned in §20 gives a necessary and sufficient condition for the relation

$$\lim_{n\to\infty} L_n(f; A) = f(z)$$

to hold, for f analytic on l.

What happens if we assume only continuity of f, and if l is "not very smooth," is a different question. (The problem when l is a broken line consisting of two segments was mentioned to me by D. J. Newman.) If

$$w = \varphi(z) \tag{40.1}$$

is the analytic function mapping one-to-one the outside of l onto $|w| > 1$, then it is natural to choose the knots $z_{\nu n}$ so that

$$\varphi(z_{\nu n}) = \exp\frac{(2\nu - 1)\pi i}{2n}, \qquad \nu = 1, 2, \ldots, n;\ n = 1, 2, \ldots. \tag{40.2}$$

The real difficulties and deviations from the case of the interval $[-1, +1]$ will be more clearly understood if we take for l the curve

$$l_1 \overset{\text{def}}{=} K_1 \cup K_2, \tag{40.3}$$

where

$$\begin{aligned} K_1 &: |z - \tfrac{1}{2}| = \tfrac{1}{2}, \qquad \operatorname{Im} z \leqslant 0; \\ K_2 &: |z + \tfrac{1}{2}| = \tfrac{1}{2}, \qquad \operatorname{Im} z \geqslant 0. \end{aligned} \tag{40.4}$$

It can easily be verified that, in this case,

$$\varphi(z) = \tan\frac{\pi}{4z}.$$

Thus, the knots are given by

$$\tan\frac{\pi}{4z_{\nu n}} = \exp\frac{(2\nu - 1)\pi i}{2n}, \qquad \nu = 1, 2, \ldots, n;\ n = 1, 2, \ldots. \tag{40.5}$$

I think that the matrix defined in this way corresponds to T. A theorem corresponding to (4.6) and (7.1)–(7.2) would follow by the solution of the following

PROBLEM XLII. *If the elements of A are on l_1 defined by* (40.3)–(40.4), *then the minimum of*

$$\max_{z \in l_1} \sum_{\nu=1}^{n} |l_{\nu n}(z; A)| \tag{40.6}$$

with respect to A is asymptotically taken for the matrix defined by (40.5). *What is its value?*

Of real interest are problems corresponding to specific choices of the curve l. More specifically, we would like to know how the singularities of l influence the approximation by polynomials on this curve. This line of thought raises also the question what is the "correct" definition of modulus of continuity. Should we define it (for rectifiable l) by

$$\omega_1(\delta, f) = \max |f(x') - f(x')|, \tag{40.7}$$

where x' and x'' are on l and their distance *measured on* l is $\leqslant \delta$, or by

$$\omega_2(\delta, f) = \max_{\substack{x', x'' \in l \\ |x' - x''| \leqslant \delta}} |f(x') - f(x'')|\,? \tag{40.8}$$

It is likely that $\omega_1(\delta, f)$ is the correct one. If so, how about non-rectifiable curves l? Hence,

PROBLEM XLIII. *What modulus of continuity should be used in the analogs of the theorems of Jackson, S. Bernstein and Müntz–Szász for curves* l *with singularities?*

For recent developments, compare Freud and Vértesi [26] and Kis and Vértesi [35] and the abstract of E. D. Lesley in the November, 1975 issue of the *Notices of the American Mathematical Society*.)

It is very likely that if l is continuously differentiable or satisfies some even stronger conditions, then the whole classical theory of approximation can be extended to it. Furthermore, it is very probable that such questions have already appeared in the literature. Therefore, I do not formulate them as open problems.

41. The case where l is closed, especially when l is the unit circle, has been the subject of many investigations. Here the role of the class $C[-1, -1]$ is played by $C[|z| \leqslant 1]$ whose elements $f(z)$ are regular in $|z| < 1$ and continuous in $|z| \leqslant 1$. The elements of the matrix A are on $|z| = 1$. Although it is clear that Lagrange interpolation is not good in general, it is still possible that the question corresponding to Problem I has an elegant solution.

PROBLEM XLIV (A conjecture of Erdös). *Is it true that if the elements of* A *are on the unit circle, then*

$$\min_A \max_{|z|=1} \sum_{\nu=1}^{n} |l_{\nu n}(z)| \tag{41.1}$$

is attained if the knots are the vertices of a regular n*-gon*? (*We denote such a matrix by* A_0 .).

In the case of $[-1, +1]$, switching from Lagrange interpolation to Hermite interpolation has helped to achieve convergence. In the case of the unit circle, this does not help. According to a remark of Kövári, none of the processes used so far is always convergent. Thus we have the following

PROBLEM XLV (Kövári). *Does there exist an interpolation process which converges for every $f \in C[|z| \leqslant 1]$, uniformly in $|z| \leqslant 1$? (A result in this direction can be found in Szabados* [52].)

One could think that the case $|z| \leqslant 1$ is always "worse" than that of $[-1, +1]$. However, this is not always so. For instance, in contrast to conjecture (33.4) which is supported to some extent by (33.2) and (33.3), O. Kis showed that, for the fundamental functions of the first kind $r_{\nu n}(z; A_0)$ of (0, 2)-interpolation,

$$\max_{|z| \leqslant 1} \sum_{\nu=1}^{n} |r_{\nu n}(z; A_0)| \leqslant c \log n, \tag{41.2}$$

which is essentially better than (33.3). It is probably simple to give a lower estimate for the left-hand side of (41.2).

Although it does not look difficult, it seems worthwhile to investigate the following

PROBLEM XLVI. *Is it true that, for all $f \in C[|z| \leqslant 1]$,*

$$\lim_{n\to\infty} \int_{|z|=1} \left| f(z) - \sum_{\nu=1}^{n} f(e^{2\pi i\nu/n})\, r_{\nu n}(z; A_0) \right|^2 |dz| = 0?$$

42. A different and interesting question (in its simplest form) is how the function e^x can be approximated by polynomials on the entire real axis. Our next problem concerns this question.

PROBLEM XLVII. *What is the smallest $a = a(n)$ such that*

$$\max_{-a(n) \leqslant x \leqslant a(n)} |e^x - \pi_n(x)| \geqslant 1$$

for every polynomial $\pi_n(x)$ of degree $\leqslant n$?

Denote by c_0 the positive root of the equation

$$\exp(1 - x^2)^{1/2} = \frac{1 + (1 + x^2)^{1/2}}{x}$$

($0.66 < c_0 < 0.67$). Révész [43] showed that, for $n \geqslant n_0(\epsilon)$, the value of $a(n)$ is between

$$c_0 n - 2c_0 n^{\epsilon-1} \qquad \text{and} \qquad c_0 n + \left(\frac{c_0}{2} + \epsilon\right) \log n.$$

V. Orthogonal Polynomials

43. In §5 I have mentioned orthogonal polynomials and their essential role in the theory of interpolation. In the general theory of approximation by polynomials, their significance can be illustrated by the fact that if $p(x)$ is a given weight function, and

$$\int_{-1}^{1} f^2(x)\, p(x)\, dx$$

exists, then the minimum of $\int_{-1}^{+1} |f(x) - \pi_n(x)|^2\, p(x)\, dx$ is taken on by the polynomial $\pi_n(x)$ which is the nth partial sum of the expansion of f in the orthogonal polynomials $q_0(x)$, $q_1(x)$,.... corresponding to the weight function $p(x)$. S. Bernstein (globally) and G. Szegö (locally) gave asymptotic representations for $q_n(x)$ under certain assumptions on $p(x)$. For many purposes these beautiful formulas are "too strong," and weaker conclusions would be sufficient. On the other hand, we would need such a weaker conclusion under essential relaxation of the conditions on $p(x)$. In this connection I mention a 50-year-old conjecture by Steklov.

Problem XLVIII (Steklov). *Let $p(x)$ satisfy* (12.6). *Is it true that, for the polynomials $q_n(x)$, orthonormalized on* $[-1, +1]$ *with weight $p(x)$, we have in* $[-1 + \epsilon, 1 - \epsilon]$ the inequality

$$|\, q_n(x)| \leqslant c(p, \epsilon), \tag{43.1}$$

independently of n?

Related to this is the following

Problem XLIX. *Is it true that, if*

$$p(x) \geqslant (1 - x^2)^{-1/2}, \tag{43.2}$$

then

$$|\, q_n(x)| \leqslant c(p), \qquad n = 1, 2, \ldots, \tag{43.3}$$

uniformly in $[-1, +1]$?

There is great interest in this problem, due to the following fact. As mentioned in §8, Freud proved (8.4) for certain matrices P under an assumption on $p(x)$ which cannot be easily checked. This assumption is just (43.3). Hence an affirmative answer to the last problem would also yield (8.4) under the condition (43.2).

44. The asymptotic formulas of Szegö and Bernstein are of the form

$$q_n(\cos\theta_0) = (1 + o(1))\left(\frac{2}{\pi}\right)^{1/2} \frac{\cos(n\theta_0 + \psi(\theta_0))}{(p(\cos\theta_0)\sin\theta_0)^{1/2}}. \tag{44.1}$$

Here θ_0 is a fixed number satisfying $\epsilon \leqslant \theta_0 \leqslant \pi - \epsilon$. In this formula $n \to \infty$. The function $\psi(\theta)$ is determined by $p(x)$. Formula (44.1) holds if, putting

$$p(\cos\theta)\sin\theta \overset{\text{def}}{=} p_1(\theta), \tag{44.2}$$

one has the relation

$$|p_1(\theta + h) - p_1(\theta)| < c\log^{-1-\delta}\frac{1}{|h|}. \tag{44.3}$$

Condition (44.3) is sufficient for (44.1). As far as I know, the question of whether or not it can be replaced by a weaker one is still open.

PROBLEM L. *Does there exist a weight function $p(x)$ for which*

$$\begin{aligned} &p(x)(1 - x^2)^{1/2} \in C[-1, +1],\\ &p(x)(1 - x^2)^{1/2} \geqslant m > 0, \end{aligned} \tag{44.4}$$

and for which, with some θ_0 $(0 < \theta_0 < \pi)$, the orthogonal polynomials q_n do not obey any asymptotic formula of type (44.1)?

To illustrate the difficulty of the problem, I mention that, in my paper with Erdös mentioned in §20, we showed that, if (44.3) holds, then, using the notation (20.6), we have, for

$$\epsilon \leqslant \theta_{\nu n} < \theta_{\nu+1,n} \leqslant \pi - \epsilon,$$

the relation

$$\lim_{n\to\infty} n(\theta_{\nu+1,n} - \theta_{\nu n}) = \pi.$$

45. In 1938 Erdös and I [10] showed that, if the integrals

$$\int_{-1}^{1} p(x)\,dx \qquad \text{and} \qquad \int_{-1}^{1} \frac{dx}{p(x)} \tag{45.1}$$

exist, then writing the zeros of the orthgonal polynomials as $\cos \theta_{\nu,n}$, we have

$$0 < \theta_{\nu+1,n} - \theta_{\nu n} \leqslant c(p) \frac{\log (n+1)}{n}. \tag{45.2}$$

It is natural to ask

PROBLEM LI. *Can the upper estimate* (45.2) *be improved*?

In our paper, we obtained (45.2) as a corollary of a more general theorem which as we showed by a counter-example, cannot be improved. However, we do not have such a counter-example for zeros of orthogonal polynomials.

46. Consider now the orthogonal polynomials $q_n(x)$ belonging to the weight function $p(x)$. We assume they are normalized as

$$q_n(x) = x^n + \cdots. \tag{46.1}$$

It is known that the recursion formula

$$xq_n(x) = q_{n+1}(x) + B_n q_n(x) + C_n q_{n-1}(x) \tag{46.2}$$

holds, where

$$C_n > 0. \tag{46.3}$$

It was an important discovery by Favard, that, conversely, any sequence of polynomials satisfying (46.2)–(46.3) is orthogonal with respect to some weight $d\alpha(x)$. About this weight function very little is known. I know only of some results of Chihara who drew conclusions from the behavior of the coefficients B_n and C_n on the behavior of $\alpha(x)$. Many years ago I suggested as a problem for the Schweitzer competition, proof of the formula

$$\lim_{n\to\infty} \left(\frac{z}{n}\right)^n K_n\left(\frac{n}{2z}\right) = e^{-z^2}, \tag{46.4}$$

where $K_n(t)$ is the nth Hermite polynomial defined in (5.6) with the normalization

$$\int_{-\infty}^{\infty} K_n(t)^2 e^{-t^2} jdt = (\pi)^{1/2} 2^n n\,! \tag{46.5}$$

The interest in this formula lies in the fact that the weight function is reproduced in a simple way by the orthogonal polynomials, for real z's. It would be desirable to be able to recover the weight function in such a way for a broader class of such functions.

PROBLEM LII. *Is there a formula for the Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$, *analogous to* (46.4)? *Or, is it true that*

$$\lim_{n\to\infty} \left(\frac{x}{n}\right) \; q_n \left(\frac{n}{2x}\right) = p(x) \tag{46.6}$$

implies $p(x) = e^{-x^2}$?

47. Formula (46.4) is interesting also in another respect. For fixed $0 < a < b$ it gives an asymptotic representation of $K_n(z)$ in the annulus $an \leqslant |z| \leqslant bn$, namely,

$$K_n(z) = (1 + o(1))(2z)^n \, e^{-n^2/4z^2}. \tag{47.1}$$

The asymptotic behavior of the $K_n(z)$'s is treated in great detail by Szegö [56]. Let us denote their zeros by

$$x_{1n} > x_{2n} > \cdots > x_{nn} . \tag{47.2}$$

Then we have, as $n \to \infty$,

$$x_{1n} = (1 + o(1))(2n)^{1/2}. \tag{47.3}$$

We can divide the asymptotic formulas into two large classes.

(a) "Outer" asymptotic formulas, valid off the real axis.

(b) "Inner" asymptotic formulas, valid on the real axis.

Since all the zeros are real, the second class of asymptotics is more interesting. Within this classification of asymptotic formulas, we have further subclasses. One of them pertains to the domain $|z| \leqslant R$, R independent of n, another to the constraint

$$|z| \leqslant c(n)^{1/2}, \tag{47.4}$$

c being a constant. (Note that none of these classes pertains to the domain of (47.1).)

Because of (47.3), the second type is more interesting because it gives information on the oscillatory behavior of the polynomials.

48. The Hermite polynomials have received much attention in the literature. One of the reasons for this interest is that it was hoped that information about these polynomials (and about the Laguerre polynomials) would lead to a general asymptotic formula for polynomials orthogonal on an infinite interval. Very little of these hopes has materialized so far. The first task should be to find the "fine" domains, that is, to solve the following

PROBLEM LIII. *Suppose that, for the weight function $p(x) > 0$, $p(x) \in L(-\infty, \infty)$, the moments satisfy*

$$\int_{-\infty}^{\infty} |x|^n p(x)\,dx < \infty, \qquad n = 0, 1, \ldots. \tag{48.1}$$

What asymptotic representation can be given for

$$\text{(a)} \quad x_{1n} = \xi(n, p), \qquad \text{(b)} \quad x_{nn} = \eta(n, p)? \tag{48.2}$$

This problem is interesting even for subclasses of weight functions.

It is easy to see that $\max_\nu |x_{\nu n}|$ tends to infinity as $n \to \infty$. In 1960, I thought that I could construct a $p(x)$ satisfying (48.1) for which $x_{nn} > -c$, that is, a $p(x)$ for which,

$$\xi(n, p) \to \infty, \qquad \eta(n, p) > -c. \tag{48.3}$$

From my notes of that time, I am unable to make a valid reconstruction. Hence I propose

PROBLEM LIV. *Does there exist a weight function $p(x)$ for which* (48.3) *holds?*

49. The asymptotic formulas on Hermite polynomials mentioned in §47 indicate the character of such formulas to be expected for polynomials $q_n(x)$ corresponding to a $p(x)$ satisfying (48.1). Regarding outer asymptotic formulas, first for $[-1, +1]$, I mention here two results. The first one is a theorem of Szegö, valid for $p(x)$ satisfying

$$p(x) \geqslant 0, \qquad p(x) \in L[-1, +1], \qquad \log^+ \frac{1}{p(x)} \in L[-1, +1]. \tag{49.1}$$

This theorem (stated for orthonormal polynomials) can be found in [56, pp. 296–297]. In my paper with Erdös mentioned in §20, we assume only that

$$p(x) \geqslant 0, \qquad p(x) \in L[-1, +1] \tag{49.2}$$

and

$$p(x) > 0 \qquad \text{a.e.} \tag{49.3}$$

However, we do not get an asymptotic formula for $q_n(x)$. What we do obtain is

$$q_n(x)^{1/n} = (1 + o(1))\,\frac{x + (x^2 - 1)^{1/2}}{2}, \tag{49.4}$$

uniformly in any bounded domain in the plane cut along $[-1, +1]$. We assume there that the highest coefficient in $q_n(x)$ is 1. The proper choice of the branch of the nth root is obvious.

PROBLEM LV. *Does there exist, for $p(x)$ with non-compact support, an asymptotic representation of type* (49.4)?

The following is basic.

PROBLEM LVI. *For which general class of weight functions $p(x)$ satisfying* (48.1) *is there an asymptotic formula for $q_n(x)$, valid in every bounded closed domain lying in* $\operatorname{Im} x > 0$?

Perhaps it is possible to get such a result from the theorem of Szegö mentioned earlier, by an appropriate passage to the limit.

50. As already mentioned in §47, the really deep questions concern "inner" asymptotics. For instance, it would be interesting to determine the behavior of the orthogonal polynomials in the interval

$$2\eta(n, p) \leqq x \leqq 2\xi(n, p). \tag{50.1}$$

Here we use the notation (48.2). Since, at present, there are no general theorems on Problem LIII, I do not state this question as a numbered problem.

PROBLEM LVII. *Find a subclass of weight functions $p(x)$ satisfying* (48.1), *for which the corresponding orthogonal polynomials have an asymptotic representation for $-a \leqslant x \leqslant a$, with arbitrarily large a.*

We have somewhat easier questions when we investigate the distribution of the zeros in the interval $(\eta(n, p), \xi(n, p))$. The only known result in this direction is due to Erdös [16]. Its statement, in qualitative form, is that, if $p(x)$ decreases "very rapidly" as $x \to \pm\infty$, then after transforming the interval (x_{nn}, x_{1n}) linearly into $(-1, 1)$, the zeros are uniformly distributed in the sense of §20 on the semicircle over $(-1, 1)$. In the case of Hermite polynomials this is not true. Thus we are naturally led to

PROBLEM LVIII. *Find a class of weight functions $p(x)$ satisfying* (48.1) *which is larger than that of Erdös, and for which we have uniform distribution of the zeros of $q_n(x)$ on the semicircle mentioned above.*

51. A seemingly easier question concerns the Cotes numbers

$$\lambda_{\nu n}(p) \overset{\text{def}}{=} \int_{-\infty}^{\infty} l_{\nu n}(x)\, p(x)\, dx, \qquad \nu = 1,\ldots, n;\ n = 1, 2,\ldots \tag{51.1}$$

For the interval $[-1, 1]$, I showed with Erdös in our paper mentioned in §20 that, if (44.4) holds, then, for all ν's satisfying

$$-1 + \frac{\log n}{n^2} \leqq x_{\nu n} \leqq 1 - \frac{\log n}{n^2}, \tag{51.2}$$

we have

$$\lambda_{\nu n}(p) = (1 + o(1)) \frac{\pi}{n} p(x_{\nu n})(1 - x_{\nu n}^2)^{1/2} \tag{51.3}$$

uniformly in ν. This result suggests the following

PROBLEM LIX. *Give a subclass of weight functions satisfying* (48.1) *for which there is an asymptotic formula of the type* (51.3).

The method used in our paper may be a good starting point in solving this important problem.

52. The problem of asymptotic representation can be treated very well in the case of weight functions whose orthogonal polynomials satisfy a differential equation or have a "handy" generating function. The recursive formula for the Hermite polynomials $K_n(x)$ of (5.6) is

$$K_n(x) = 2xK_{n-1}(x) - 2(n-1)\, K_{n-2}(x). \tag{52.1}$$

Assume that, instead of this relation, we have

$$q_n(x) = (ax + b)\, q_{n-1}(x) - (cn^2 + dn + e)\, q_{n-2}(x), \tag{52.2}$$

where a, b, c, d and e are numerical constants satisfying

$$a > 0, \qquad cx^2 + dx + e \geqq 0, \qquad \text{for} \quad x \geqq 2. \qquad (52.))$$

Then, according to Favard's theorem, $q_n(x)$ is a sequence of orthogonal polynomials corresponding to some weight $d\alpha(x)$. If we consider x as a parameter, the function

$$F(z; x) = \sum_{n=0}^{\infty} q_n(x)\, e^{-nz} \tag{52.4}$$

satisfies a differential equation of the second order in z. This enables us to investigate the behavior of $F(z, x)$ and $q_n(z)$ by means of complex function theory. Because of continuity reasons, one can expect that, with an appropriate choice of a, b, c, d and e, the weight function must be positive on the whole real axis. In this way one could increase the class of weight functions for which there is an inner asymptotic. The same holds if we replace the constant b in (52.2) by an appropriate polynomial of second degree in n.

PROBLEM LX. *Investigate the inner asymptotics for the orthogonal polynomials belonging to the above generalizations of* (52.1).

The study of even more general recursions seems also possible.

53. Szegö and Carleman introduced orthogonal polynomials in a broader sense. (See Szegö [56, pp. 364–366].) I shall not consider here the most general case. Let l be a rectifiable Jordan curve. If $p(\xi) \geqslant 0$ is defined for $\xi \in l$, we say that the polynomials $\varphi_n(z) = \varphi_n(z; l, p)$ are orthogonal in Szegö's sense if the relations

$$\begin{aligned} \int_l \varphi_n(\xi)(\bar{\xi})^\nu p(\xi) |d\xi| &= 0, \qquad \text{for } \nu < n, \\ \frac{1}{|l|} \int_l |\varphi_n(\xi)|^2 p(\xi) |d\xi| &= 1 \end{aligned} \tag{53.1}$$

hold, $|l|$ being the length of l. Carleman replaced the assumption of rectifiability by a weaker one. Namely, he replaced the line integral along l by the double integral over the interior of l. Of course, the weight function is defined, in that case, in that domain. These polynomials are important because they are closely connected with the function $\Phi(z)$ which maps the outside of l one-to-one onto $|w| > 1$. In fact,

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = \Phi(z),$$

for every z exterior to l. For his polynomials, Szegö developed outer and inner asymptotics, the latter under rather strong conditions on l. It is a natural task to weaken them. Results in this direction, which are probably improvable, can be found in D. Gaier's monograph "Konstruktive Method der konformen Abbildung," p. 136. I shall not formulate such problems explicitly. I state here only the following related questions.

PROBLEM LXI. *Let l be a rectifiable Jordan curve. Is there an elegant direct relation between Szegö's orthonormal polynomials and Carleman's, perhaps with appropriate weight functions?*

PROBLEM LXII. *Is there an inequality connecting the two kinds of polynomials?*

PROBLEM LXIII. *If the domain enclosed by l varies, how do the orthogonal polynomials change?*

54. A slightly different problem in the same area is the following

PROBLEM LXIV (Szegö and Walsh). *Find conditions on a sequence of Jordan curves $l_1, l_2, \ldots, l_\nu$ quaranteeing that the polynomials $\{\varphi_n(z)\}$ are orthogonal on every l_j with some weight function $p_j(z)$, where $p_j(z)$ $(j = 1, 2, \ldots, \nu)$ are Lebesgue-integrable and $\geqslant 0$ on l_j.*

Regarding the literature on the question, see Merriman [39] and Szegö's paper [55] simplifying Merriman's work.

55. The fact that, for $|z| = 1$ and $p(z) \equiv 1$, the powers z^n are orthogonal, calls attention to the essential difference between polynomials orthogonal on an interval and those orthogonal on the circle. While the zeros of polynomials orthogonal on an interval with respect to some Lebesgue-integrable weight function are simple, this is not the case for the circle.

PROBLEM LXV. *Characterize the Jordan arcs or Jordan curves l for which the zeros of the orthogonal polynomials with respect to every Lebesgue-integrable weight function on l are simple.*

It is not impossible that the only such arcs are finite or infinite intervals. For such arcs or curves, one can form the Lagrange interpolation polynomials.

The following problem does not seem to be easy.

PROBLEM LXVI. *It is known that the zeros of the nth orthogonal polynomial (with respect to a Lebesgue-integrable function on an interval) separate the zeros of the $(n + 1)$th polynomial. What corresponds to this fact on the circle?*

The zeros of orthogonal polynomials on $|z| = 1$ with respect to different weight functions have varying characters. If $z = e^{i\theta}$ and $p(\theta) \equiv 1$, then the zeros of the orthogonal polynomials are all at $z = 0$. On the othr hand, if

$$p(\theta) = \frac{1}{4}\,|1 - e^{i\theta}|^2 = \cos^2\frac{\theta}{2}, \tag{55.1}$$

then, as is easily verified,

$$q_n(z) = 1 + 2z + \cdots + (n + 1)\,z^n. \tag{55.2}$$

The zeros of these polynomials are all simple, lie in $|z| < 1$, and approach the circle $|z| = 1$ uniformly, as $n \to \infty$. They are also very uniformly distributed in each angular domain $\alpha \leqq \arg z \leqq \beta$.

Let us call these two types of weight functions, first and second types, respectively. Weight functions of the third type are those for which the zeros of the corresponding orthogonal polynomials are everywhere dense in $|z| < 1$.

PROBLEM LXVII. *Do weight functions of the third type exist?*

Instead of Jordan curves, we formulate the general question for the circle only.

PROBLEM LXVIII. *Find a class of weight functions $p(\theta)$ on the circle $z = e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$ for which the number of zeros of the corresponding orthogonal polynomials $q_n(z, p)$ in each given Jordan measurable domain in $|z| \leqq 1$ obeys an asymptotic distribution law as $n \to \infty$.*

56. Although the next problem is much easier, it is still, in some sense, very interesting. As far as I know, in the theory of complex interpolation, the knots are always chosen to lie on the Jordan curve in question, and our aim is to approximate functions belonging to a certain class, defined on the closed interior of the curve. We have a different situation if, for instance, $f(z) \in \mathrm{Lip}_\alpha(|z| \leqq 1)$ and the interpolation knots are the zeros of the polynomials (55.1)–(55.2). (The functions in the class $\mathrm{Lip}_\alpha(|z| \leqq 1)$ are regular in $|z| < 1$ and satisfy the inequality $|f(z_1) - f(z_2)| \leqslant M|z_1 - z_2|^\alpha$ for $|z_1|, |z_2| \leqslant 1$.) So a simple form of a general problem is

PROBLEM LXIX. *For which class $\mathrm{Lip}_\alpha(|z| \leqq 1)$ does the Lagrange interpolation at the above knots converge uniformly in $|z| \leqq 1$?*

A more general question would be to replace the polynomials (55.1)–(55.2) by orthogonal polynomials corresponding to a general weight function of the second type. But I shall not state it as a separate problem.

57. The Hermite polynomials are important for yet another reason. We can obtain bounds for the roots of the equation

$$a_0 + a_1 z + \cdots + a_n z^n = 0 \tag{57.1}$$

in terms of the coefficients. For some questions, it is more important to get strips along the real axis which contain at least one root of (57.1). I men-mentioned in my lecture "Sur l'algebre fonctionnelle" at the First Hungarian Mathematical Congress that, for this purpose, one should write the polynomial in the form

$$b_0 K_0(z) + \cdots + b_n K_n(z), \tag{57.2}$$

where $K_n(z)$ are the Hermite polynomials normalized by (46.5). I mention here only one result in this direction due to Makai and myself [36]. One can find further developments in the encyclopedia article [45] by Specht. Our result in [36] asserts that any "trinomial" equation

$$K_0(z) + K_1(z) + bK_n(z) = 0 \tag{57.3}$$

has a zero in the strip

$$|\operatorname{Im} z| \leqq c, \tag{57.4}$$

c being an absolute constant. Later, Schmeisser showed that the exact value of c is $\frac{1}{2}$.

A natural question is

PROBLEM LXX. *Is there a constant c_1 such that any equation*

$$K_0(z) + K_1(z) + b_1K_{n_1}(z) + b_2K_{n_2}(z) = 0 \tag{57.5}$$

has a zero in the strip

$$|\operatorname{Im} z| \leqq c_1 \;? \tag{57.6}$$

58. We return to the case of the interval $[-1, +1]$. Let $q_n(x)$ be the nth orthogonal polynomial corresponding to the weight function $p(x)$. As I have shown in my paper [62], for

$$-1 \leqq b - \delta < b + \delta \leqq +1, \tag{58.1}$$

we have the relation

$$\lim_{n\to\infty} \int_{b-\delta}^{b+\delta} q_n(x)^2\, p(x)\, dx = \frac{1}{\pi} \int_{b-\delta}^{b+\delta} \frac{dx}{(1-x^2)^{1/2}}, \tag{58.2}$$

for each integrable $p(x)$ that satisfies $p(x) \geqslant 0$ and

$$\log^+ \frac{1}{p(x)} \in L[-1, +1]. \tag{58.3}$$

Hence we have, for $n > n_0(\delta, p)$,

$$\int_{b-\delta}^{b+\delta} q_n^2(x)\, p(x)\, dx > \frac{1}{2\pi} \int_{b-\delta}^{b+\delta} \frac{dx}{(1-x^2)^{1/2}} > \frac{\delta}{\pi}. \tag{58.4}$$

Here, however, $n_0(\delta, p)$ is ineffective, that is, it cannot be calculated explicitly. Because of a reason to be explained in the next section, we need an explicit $n_0(\delta, p)$.

PROBLEM LXXI. *Give an explicit estimate for* $n_0(\delta, p)$ *such that, if* (58.3) *holds, then so does* (58.4), *for* $n > n_0(\delta, p)$.

59. The background of the last problem is a theorem of N. Wiener which asserts (in its improvement by Ingham) that, if $0 < \epsilon < 1$, $0 < \delta < 1$, and

$$f(t) = \sum_{j=1}^{N} a_j \cos \nu_j t, \tag{59.1}$$

where $\nu_1, \nu_2, \ldots, \nu_N$ are natural integers satisfying the gap condition

$$\nu_{j+1} - \nu_j \geqslant \frac{\pi + \epsilon}{\delta}, \tag{59.2}$$

and a_j are arbitrary complex numbers, then, with an effective $c(\epsilon)$, we have

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt \leqq \frac{c(\epsilon)}{2\delta} \int_{b-\delta}^{b+\delta} |f(t)|^2 \, dt, \tag{59.3}$$

independently of N, b and the coefficients a_j. Putting $\cos t = x$, (59.3) transforms into an inequality of the type

$$\int_{-1}^{1} \frac{|g(x)|^2}{(1 - x^2)^{1/2}} \, dx \leqq c(\epsilon, \delta) \int_{b-\delta}^{b-\delta} \frac{|g(x)|^2}{(1 - x^2)^{1/2}} \, dx, \tag{59.4}$$

where (with the notation (5.5))

$$g(x) = \sum_{j=1}^{N} a_j T_{\nu_j}(x)$$

and the gap condition (59.2) holds. It is natural to ask whether the weight function $(1 - x^2)^{-1/2}$ could be replaced by one belonging to a general class. In other words, we ask if it is true that for

$$G(x) = \sum_{j=1}^{N} b_j q_{\nu_j}(x) \qquad (0 \leqq \nu_1 < \cdots < \nu_N), \tag{59.5}$$

we have the inequality

$$\int_{-1}^{1} |G(x)|^2 p(x) \, dx \leqq c_1(\delta, p) \int_{b-\delta}^{b+\delta} |G(x)|^2 p(x) \, dx \tag{59.6}$$

independently of b_j, N and b, if only

$$-1 \leqq b - \delta < b + \delta \leqq 1, \tag{59.7}$$

and if a gap condition

$$\nu_1 \geqslant B(\delta, p), \qquad \nu_{j+1} - \nu_j \geqslant B(\delta, p), \qquad j = 1, 2, \ldots, N-1 \quad (59.8)$$

is satisfied, with a suitable $B(\delta, p)$.

In my paper mentioned in the previous section I showed that this is true for a surprisingly broad class of weight functions, namely, for the class of weight functions satisfying (58.3). Since $n_0(p, \delta)$ was not given explicitly, only the existence of a $B(\delta, p)$ was shown. To give $B(\delta, p)$ explicitly, we would need the solution of last problem.

60. Inequalities (59.6), (59.7) and (59.8) have an interesting connection with the theory of polynomial approximation. The theorem of Müntz and Szász, mentioned in connection with Problem XLIII, states, that, if

$$0 = m_0 < m_1 < \cdots < m_n < \cdots \qquad (60.1)$$

are integers satisfying

$$\sum_{j=1}^{\infty} \frac{1}{m_j} = \infty, \qquad (60.2)$$

then, for every $f(x) \in C[0, 1]$, and every $\epsilon > 0$, there is a polynomial $\sum_{j=0}^{\infty} b_j x^{m_j}$ such that

$$\max_{0 \leqslant x \leqslant 1} \left| f(x) - \sum_{j=0}^{N} b_j x^{m_j} \right| \leqq \epsilon. \qquad (60.3)$$

In other words, in the theorem of Weierstrass, we do not need all integral, non-negative powers of x. Instead, it suffices to take a subset satisfying (60.1) and (60.2). We can choose $m_0, m_1, \ldots, m_n$ so that there are arbitrarily large gaps; even $m_{j+1} - m_j \to \infty$ can hold. For instance, we can take $m_j = [j \log j]$. If we replace $\{x^m\}$ by the system of orthogonal polynomials corresponding to a weight function $p(x)$ (which is advantageous for some purposes), and replace the interval [0, 1] by [-1, -1], then, as known, there is no theorem of Müntz–Szász type. We cannot drop a single term from the sequence $\{q_n(x)\}$. On the other hand, for $p(x) = (1 - x^2)^{-1/2}$, and $f(x) \in C[0, 1]$, $f(x)$ can be approximated arbitrarily close linear combinations of $T_{2\nu}(x)$. The general question, which seems to be very difficult, is the following

PROBLEM LXXII. *Let $\{q_\nu(x)\}$ be the sequence of orthogonal polynomials on $[-1, +1]$ corresponding to a weight function $p(x)$. Further, let $[a, b]$ be a proper subinterval of $[-1, +1]$. Characterize the non-negative integers $k_1 < \cdots < k_\nu < \cdots$ such that linear combinations of $q_{k_\nu}(x)$ can approximate, arbitrarily close, every continuous function in $[a, b]$.*

Even the following weaker form of this problem seems to be interesting and difficult.

PROBLEM LXXIII. *Suppose that $p(x)$ satisfies condition* (58.3). *Is it true that, for every proper subinterval $[a, b]$ of $[-1, +1]$, there is a $D = D(a, b, p)$ such that $0 < D < 1$, and every subsequence $\{q_{k_\nu}(x)\}$ has dense finite linear combinations in the space of functions continuous in $[a, b]$, if only the lower density of k_ν is greater than D?*

61. In §26 we discussed polynomials minimizing

$$\int_{-1}^{1} \pi_n(x)^4 p(x)\, dx. \tag{61.1}$$

(I restrict myself to the case $m = 3$ in (26.5).) The next two problems are connected with that topic.

PROBLEM LXXIV. *Give the minimizing polynomials in an explicit form, for weight functions other than $(1 - x^2)^{-1/2}$.*

PROBLEM LXXV. *Give an asymptotic representation of the minimizing polynomials, valid on $[-1, +1]$, for a weight function other than $(1 - x^2)^{-1/2}$.*

Results in this direction can be found in my paper [12] with Erdös, and in [22] by Frenkel–Fertig.

VI. RATIONAL APPROXIMATION

62. The polynomials form a linear set. It is natural to ask what are the basic problems in the non-linear theory of approximation. The simplest problem of this kind is that of uniform approximation of the elements of $C[-1, +1]$ by rational functions, that is, by functions of the form

$$R_n(x) = \frac{\pi_n^*(x)}{\pi_n^{**}(x)}, \tag{62.1}$$

where $\pi_n^*(x)$ and $\pi_n^{**}(x)$ are polynomials of degree $\leqslant n$. Besides the problem of approximation by polynomials, Chebyshev was already interested in the theory of rational approximation. It is peculiar that, while the theory of polynomial approximation has had an extensive, growing literature, approximation by rational functions in the real domain did not get any attention from 1908 until about 15 years ago. The reason for this is probably the fact that for the elements of the classes $\mathrm{Lip}_\alpha[-1, +1]$, which served in the theory of polynomial approximation as "test classes," approximation by functions of the form (62.1) is not better than approximation by polynomials. More exactly, by a slight modification of an old example of S. Bernstein, it is easy to see that

$$f_0(x) = \sum_{\nu=1}^{\infty} \frac{1}{\nu!^\alpha} T_{\nu!}\left(\frac{x}{2}\right) \tag{62.2}$$

belongs to $\mathrm{Lip}_\alpha[-1, +1]$ for every $0 < \alpha < 1$. On the other hand, we have for any $R_n(x)$,

$$\max_{-1 \leqslant x \leqslant +1} |f_0(x) - R_n(x)| \geqslant \frac{1}{n^\alpha \log^3 n}. \tag{62.3}$$

(I know this example from a letter of D. J. Newman.)

According to the theorem of D. Jackson, the best polynomial approximation of the same function is of order $c(\alpha)\, n^{-\alpha}$. For a long time this phenomenon discouraged any hope that rational approximation can do better than polynomial approximation. Szabados [49] proved an even stronger genative result according to which, for every $0 < \alpha < 1$, there is a function $f_1(x) \in \mathrm{Lip}_\alpha$ such that, for every $R_n(x)$,

$$\max_{-1 \leqslant x \leqslant +1} |f_1(x) - R_n(x)| > c(\alpha)\, n^{-\alpha}.$$

63. A theorem of Newman of 1964 [40], according to which

$$||x| - R_n^*(x)| \leqslant e^{-c_1(n)^{1/2}}, \qquad |x| \leqslant 1, \tag{63.1}$$

for a suitable $R^*(x)$, but, for every $R_n(x)$,

$$||x| - R_n(x)|| \geqslant e^{-c_2(n)^{1/2}}, \tag{63.2}$$

was a great surprise. (Here $c_1, c_2, \ldots$ are positive constants.)

This discovery raised new hopes. It was surprising because of a result in the famous paper by Bernstein [4] of 1912, according to which, for a suitable polynomial $\pi_n^*(x)$ of degree $\leqslant n$,

$$||x| - \pi_n^*(x)| < \frac{c_3}{n} \qquad (-1 \leqslant x \leqslant +1), \tag{63.3}$$

but for every such polynomial $\pi_n(x)$,

$$\max_{-1 \leqslant x \leqslant +1} ||x| - \pi_n(x)| > \frac{c_4}{n}. \tag{63.4}$$

Newman's discovery raised hopes because of the well-known role played by the function $|x|$ in the theory of polynomial approximation. (For instance, from the approximability of $|x|$ by polynomials, one can deduce Jackson's theorem.) However, these hopes soon abated because of the observation that if $|x|$ played the same role in rational approximation as it does in the theory of polynomial approximation, then we could expect the elements of $\mathrm{Lip}_\alpha[-1, +1]$ to be better approximated by rational functions than (63.3) allows. The difference between the two kinds of approximations is that the sum of two polynomials of degree n is again a polynomial of degree n, but this is not the case for rational functions. How much the hopes abated is shown by the following problem of Newman (Intern. Series of *Numer. Math.* **5** (1964), 189) which is still open:

PROBLEM LXXVI (D. J. Newman). *Is it true that, for every function $f(x) \in \mathrm{Lip}[-1, +1]$, the rate of best approximation by rational functions of degree n is $o(1/n)$?*

It seemed that Newman's result (63.1), (63.2) is a beautiful but isolated theorem for a special $f(x)$.

64. The inequality (63.1) of Newman became of basic importance when P. Szüsz and I asked whether there are "large" classes of functions, different from $\mathrm{Lip}_\alpha[-1, +1]$, whose elements can be approximated by rational functions essentially better than by polynomials. Of the classes we obtained, I will mention only one, for which there is a particularly great contrast. This is the class Z of functions which are continuous and piecewise analytic in $[-1, +1]$. Historically, next to the class of analytic functions, this class is, perhaps, the "most classical". In general, as one can see, for instance, from (63.4), we do not have a polynomial approximation better than $O(1/n)$. On the other hand, for every $f \in Z$, there is a rational function $R_n^*(x)$ such that

$$|f(x) - R_n^*(x)| < c_1(f)\, e^{-c_2(f)(n)^{1/2}}, \tag{64.1}$$

where c_1 and c_2 depend on f but not on n. I gave a simplified proof of this inequality at the international conference on complex analysis at Erevan in September 1965 [61]. From this proof it becomes clear that, if $f(x)$ is piecewise analytic then $c_1(f)$ depends only on $\max_{|x|\leqslant 1} |f(x)|$, and c_2 only on the domains containing the intervals in which $f(x)$ is analytic. As shown by (63.4) and (64.1), the approximability by rational functions is essentially better than that by polynomials. One question still remains. By inequality (63.2) of Newman, $e^{-c(n)^{1/2}}$ is the correct order of magnitude in (64.1). On the other hand, if $f(x)$ is analytic in a domain containing $[-1, +1]$, then the order of magnitude of the (best) error term is $e^{-c_3(f)n}$ even with polynomial approximations.

PROBLEM LXXVII. *What is the "real reason" for the exponent* $n^{1/2}$ *in* (64.1)? *Why is it not, for instance,* $n^{2/3}$?

65. The first class of functions found by me and Szüsz (the first known class of functions for which rational approximation is better than polynomial approximation) was the class Z_1 of functions which are convex in $[-1, +1]$. As can be seen from (63.4), polynomial approximation need not be better than $O(1/n)$. On the other hand, we showed in [57] that, for every $f(x) \in Z_1$,

$$|f(x) - R_n^*(x)| \leqslant c_5(f, \epsilon) \frac{\log^4 n}{n^2} \tag{65.1}$$

in the interval $[-1 + \epsilon, 1 - \epsilon]$, with a suitable $R_n^*(x)$.

Inequality (65.1) was soon improved by Freud [25] who replaced our $\log^4 n$ by $\log^2 n$. However, a really extraordinary improvement was achieved by Popov [41] who showed that, for $[-1 + \epsilon, 1 - \epsilon]$, we have

$$|f(x) - R_n^*(x)| < c_6(f, \epsilon, k) \frac{\log_k n}{n^2} \tag{65.2}$$

for some $R^*(x)$. Here $\log_k n$ is the k-times iterated logarithm. We now ask:

PROBLEM LXXVIII. *Can* (65.2) *be improved to*

$$|f(x) - R_n^*(x)| < c(f, \epsilon) \frac{1}{n^2} \; ? \tag{65.3}$$

The last problem becomes even more interesting if we take into account a remark of Freud, according to which an affirmative answer to it would imply the same for Problem LXXVI.

66. I discussed in my lecture at Erevan the reasons why, for some classes of functions, rational approximation is better than polynomial approximation. The example of $|x|$ shows that polynomial approximation can be spoiled by a "bad" behavior of the approximated function at a single point. Approximation by rationals is much less sensitive. It seems that rational approximation is much less affected by a "bad" behavior of the approximated function at a finite number of points or even on a "small" infinite set. To give a "quantitative" analysis of this, it is convenient to consider the following class $Z_2 = Z_2(\alpha, \beta)$ of functions. Let

$$0 < \alpha < \beta < 1, \tag{66.1}$$

and suppose that, for $-1 \leqslant x' < x'' \leqslant 1$,

$$|f(x') - f(x'')| \leqslant |x' - x''|^{\alpha}. \tag{66.2}$$

Further, suppose that $a_1, \ldots, a_k$ are real numbers satisfying

$$1 = a_0 > a_1 > \cdots > a_{k-1} > a_k = -1, \tag{66.3}$$

and let

$$0 < \epsilon < \tfrac{1}{4} \min_{\mu}(a_{\mu} - a_{\mu-1}). \tag{66.4}$$

Assume that, for every $1 \leqslant \mu \leqslant k$ and every $a_{\mu-1} + \epsilon \leqslant \xi' < \xi'' < a_{\mu} - \epsilon$, we have an inequality of the form

$$|f(\xi') - f(\xi'')| \leqslant \psi_{\mu}(\epsilon)(\xi'' - \xi')^{\beta}. \tag{66.5}$$

The class of such functions $f(x)$ can somewhat vaguely be described as the subclass of $\mathrm{Lip}_{\alpha}[-1, +1]$ whose functions satisfy a Lipschitz condition with a larger exponent β on subintervals; the larger the subinterval is, the larger is the constant with which the condition is satisfied. It is trivial that these functions can be approximated by rational functions, even by polynomials, to the order $O(n^{-\alpha})$. In a lecture held in March 1965 at the Hungarian Academy of Sciences on my results with Szüsz [58], I stated on the basis of superficial reasoning that, if $f \in Z_2$, then, for some $R_n^*(x)$,

$$|f(x) - R_n^*(x)| < c(f)\, n^{-\beta}, \tag{66.6}$$

independently of $\psi_{\mu}(\epsilon)$. Later, as I was unable to reconstruct our reasoning, I mentioned the matter to Szabados, who proved a weaker form of our statement. He showed that, if $\psi_{\mu}(\epsilon) = \log^{\gamma}(1/\epsilon)$ (γ a constant), then, for some $R_n^*(x)$,

$$|f(x) - R_n^*(x)| < c(f) \frac{\log^{\gamma} n}{n^{\beta}} \tag{66.7}$$

(see [50]). It is still an open problem whether or not this result can be improved.

Problem LXXIX. *Let α, β and $d = \dim(a_{\mu} - a_{\mu+1})$ be fixed. What is the fastest growth of $\psi_{\mu}(\epsilon)$, allowing an inequality of the type*

$$|f(x) - R_n^*(x)| < c(\alpha, \beta, d)\, n^{-\beta} \max_{-1 \leqslant x \leqslant 1} |f(x)| \;? \tag{66.8}$$

67. In order to formulate the problem more generally, let J be a subclass of $C[-1, +1]$, and denote by $P_n(J)$ the "optimal" polynomial approximation, that is,

$$P_n(J) = \sup_{f \in J} \min_{\pi_n} \max_{-1 \leqslant x \leqslant +1} |f - \pi_n|. \tag{67.1}$$

Let $E_n(J)$ be the optimal rational approximation, that is

$$E_n(J) = \sup_{f \in J} \min_{R_n} \max_{-1 \leqslant x \leqslant +1} |f - R_n|. \tag{67.2}$$

Then we pose

PROBLEM LXXX. *Give sufficient conditions for J guaranteeing that*

$$\lim_{n \to \infty} \frac{E_n(J)}{P_n(J)} = 0.$$

68. As Freud remarked in a conversation, a convex function $f(x)$ admits a polynomial approximation to the order $O(1/n^2)$ in the L_1-metric. This observation, related to §65, suggests our following

PROBLEM LXXXI. *Let*

$$P_n^{(1)}(J) = \sup_{f \in J} \min_{\pi_n} \int_{-1}^{1} |f - \pi_n|\, dx.$$

Find subclasses J for which

$$E_n(J)/P_n^{(1)}(J)$$

remains between two positive constants, as $n \to \infty$.

69. The problem of interpolation with rational functions of degree $\leqslant n$ occurs already in the investigations of Cauchy. Nevertheless, a theory of its convergence does not yet exist. The reason for this may be the following. Put

$$R_{\mu\nu}(x) = \frac{\pi_\mu(x)}{\pi_\nu^*(x)}; \qquad \mu, \nu \geqslant 1, \tag{69.1}$$

and let $R_{\mu\nu}$ be the set of rational functions of this form. (Recall that $\pi_k(x)$ is a polynomial of degree at most k.) The values of any function of the class $R_{\mu\nu}$ can be "in general", but not always, prescribed at $\mu + \nu + 1$ points. The only "natural" way of developing a theory of convergence of interpolation by rational functions would be to take as knots of the interpolation, for instance, elements of the matrix T, and to consider those $R_{\mu\nu}(x, T)$ which coincide with the approximated function f at the zeros of $T_{\mu+\nu+1}(x)$. But it is not evident that such rational functions do exist for any pair (μ, ν). Therefor, in order to construct a theory of convergence of rational interpolation, we have to first solve several problems. I mention only one:

PROBLEM LXXXII. *Let μ, ν be given. For $\mu - \nu + 1$ variable knots $x_1, \ldots, x_{\mu+\nu+1}$, what is the maximal number $M = M(\mu, \nu)$ such that, at least M of the relations*

$$R_{\mu\nu}^{\times}(x_j) = y_j \qquad (j = 1, 2, \ldots, \mu + \nu + 1) \tag{69.2}$$

can be satisfied for any choice of y_j ?

It is trivial that $M \leqslant \mu$, because, with the choice

$$y_1 = y_2 = \cdots = y_{\mu-\nu-1} = 0, \tag{69.3}$$

no more than μ equalities can be satisfied in (69.2). Of course, zero, in (69.3), could be replaced by any other constant.

70. In §60, I mentioned the theorem of Müntz–Szász for polynomial approximation. An analogous question, raised by Newman, is a condition on the sequence of exponents m_j assuring that every continuous function in $[0, 1]$ can be approximated uniformly by rational functions having in their numerator and denominator only powers belonging to the sequence $\{m_j\}$. In contrast to (60.2) (which is also necessary), Somorjai [44] found the surprising theorem that a sufficient condition is $m_j \to \infty$, no matter how fast this takes place. On the other hand, the following is still open.

PROBLEM LXXXIII (D. J. Newman). *Find conditions on two sequences $\{m_j'\}$ and $\{m_j''\}$ assuring that every continuous functions can be approximated arbitrarily close by rational functions having in their numerator only powers belonging to $\{m_j'\}$, and in their nominator only powers belonging to $\{m_j''\}$.*

71. Making the substitution $x = e^s$, the theorem of Müntz–Ssász can be stated in terms of functions on $[-\infty, 0]$. Let $C_0[-\infty, 0]$ denote the class of continuous functions in $(-\infty, 0]$ satisfying $f(-\infty) = \lim_{s\to-\infty} f(s) = 0$. Then condition (60.2) assures that every $f \in C_0[-\infty, 0]$ can be approximated uniformly on $[-\infty, 0]$ by linear combinations of exponentials $e^{m_j s}$. Now replace the interval $[-\infty, 0]$ by a continuous curve γ joining 0 to $-\infty$ in such a way that the angle between each chord of γ and the real axis is less than $\pi/2$. Korevaar proved in 1973 that the theorem of Müntz–Szász remains true if $[-\infty, 0]$ is replaced by such a curve γ (See "Proceedings, International Symposium (Austin, 1973).") Our next problem is connected with this theorem.

PROBLEM LXXXIV. *Does Somorjai's theorem remain true if we replace the interval* $[-\infty, 0]$ *by a curve* γ *satisfying the conditions of Korevaar's theorem?*

72. The approximability by rational functions of functions $f(z) \in C[|z| \leqslant 1]$ (this is the class of functions regular in $|z| < 1$ and continuous for $|z| \leqslant 1$) was subject to a detailed investigation by Walsh and his students. The common feature in their results was that the poles of the approximating rational functions were "kept away" from $|z| \leqslant 1$, and the order of magnitude of the approximation was not essentially better than that of approximation by polynomials. In my lecture at Erevan, I stressed the fact, which may appear paradoxical at first glance, that allowing the poles to approach $[-1, +1]$ causes better approximability. I raised the question whether this can be also the case for the class $C[|z| \leqslant 1]$. The first subclass of $C[|z| \leqslant 1]$ with better rational approximation was found by Szabados [51] in 1968. A characteristic special case of his result is as follows. If $f(z)$ is regular in $|z| < 1$, and, with the exception of $z = 1$, also in the circle $|z + \delta| \leqslant 1 + \delta$ with $0 < \delta < \frac{1}{20}$, and if, further, $f(z)$ satisfies for $|z| \leqslant 1$ a Lipschitz condition with the exponent α, then

$$|f(z) - R_n^*(z)| \leqslant c(f)\left(\frac{\log n}{n}\right)^{2\alpha}, \tag{72.1}$$

for a suitable $R_n^*(x)$. For comparison, polynomial approximation would give only $O(1/n^\alpha)$. According to a remark of Newman, (72.1) could not be improved to an upper estimate sharper than $O(1/n^{2\alpha})$.

PROBLEM LXXXV (L. Leindeler). *Can* (72.1) *be improved to* $O(1/n^{2\alpha})$?

The domain of analyticity of every element $f(z)$ of Szabados's class contains the unit disk as a proper subdomain. Now denote by S the class of functions that are analytic in $|z| < 1$, and continuous in $|z| \leqslant 1$ and which cannot be continued analytically beyond $|z| = 1$.

PROBLEM LXXXVI. *Is it true that, for* $f \in S$, *we have*

$$|f(x) - R_n^*(z)| = o(1)\,\omega\left(f, \frac{1}{n}\right), \qquad |z| \leqslant 1,$$

with a suitable R_n^* ? *Here* $\omega(f, \delta)$ *denotes the modulus of continuity of* f.

If this is true, then probably it is the best possible inequality.

PROBLEM LXXXVII. *Is it true that there is no* $f_0(z) \in S$ *such that the best approximation by polynomials of degree* $\leqslant n$ *of* $f(z)$ *is* $\geqslant c_1/n$, *but the best approximation by such rational functions is* $< e^{-c_2(n)^{1/2}}$?

73. All these questions pertain to the case of one real or one complex variable. I am not familiar with the literature on approximation of functions of several real variables. There are a number of natural questions whose solutions are probably known; for instance, if $\pi_{\mu\nu}(x, y)$ is a polynomial of degree μ in x and ν in y, and if $\pi_{\mu\nu}$ is less than 1 in absolute value in a domain D of the x, y-plane, then what are the exact values of

$$\max_D \left| \frac{\partial \pi_{\mu\nu}}{\partial x} \right| \qquad \text{and} \qquad \max_D \left| \frac{\partial \pi_{\mu\nu}}{\partial y} \right| ?$$

Thus, I do not state them as open problems. On the other hand, I mention with some comments the following question, which is important for practical purposes.

Let D be a bounded closed domain in the (x, y)-plane with a smooth boundary. Let $f(x, y)$ be a function having continuous second partial derivatives in D. The values of the function are known by observations at

$$N = (\mu + 1)(\nu + 1) - 1 \tag{73.1}$$

different points of D which are denoted by $P_j = (x_j, y_j) \in D$. Let

$$\pi_{\mu\nu}(x, y) = \sum_{l_1=0}^{\mu} \sum_{l_2=0}^{\nu} c_{l_1 l_2} x^{l_1} y^{l_2} \tag{73.2}$$

be the polynomial having the property

$$\pi_{\mu\nu}(x_j, y_j) = \pi_{\mu\nu}(P_j) = f(P_j), \qquad j = 1, 2, \ldots, N. \tag{73.3}$$

Now choose the points $P_1, \ldots, P_N$ so that the determinant $\Delta(P_1, \ldots, P_N)$ of the system is maximum:

$$|\Delta(P_1, \ldots, P_N)| = \max_{Q_1, \ldots, Q_N \in D} |\Delta(Q_1, \ldots, Q_N)| . \tag{73.4}$$

It is easy to see that the maximum in (73.4) is positive. With this choice of the points P_k, the polynomials (73.2)–(73.3) are uniquely determined. Setting $P = (x, y)$, we have

$$\pi_{\mu\nu}(P) = \sum_{j=1}^{N} f(P_j) \frac{\Delta(P_1, \ldots, P_{j-1}, P, P_{j+1}, \ldots, P_N)}{\Delta(P_1, \ldots, P_N)}$$

$$\overset{\text{def}}{=} \sum_{j=1}^{N} f(P_j)\, l_j(P; P_1, \ldots, P_N).$$

Let $\pi^*_{\mu\nu}(P, f, D)$ be the best approximating polynomial to f in D with degrees $\leqslant \mu$ and $\leqslant \nu$. Since, obviously,

$$\sum_{j=1}^{N} \pi^*_{\mu\nu}(P_j)\, l_j(P, P_1, \dots, P_N) = \pi^*_{\mu\nu}(P), \tag{73.6}$$

we have

$$\pi_{\mu\nu}(P) - \pi^*_{\mu\nu}(P, f, D) = \sum_{j=1}^{N} (f(P_j) - \pi^*_{\mu\nu}(P_j)) \cdot l_j(P, P_1, \dots, P_N). \tag{73.7}$$

Let

$$\max_{P \in D} |f(P) - \pi^*_{\mu\nu}(P)| \overset{\text{def}}{=} d_{\mu\nu}(f, D). \tag{73.8}$$

Then we have

$$\begin{aligned} |\pi_{\mu\nu}(P) - f(P)| &\leqslant |\pi_{\mu\nu}(P) - \pi^*_{\mu\nu}(P)| + |\pi^*_{\mu\nu}(P) - f(P)| \\ &\leqslant d_{\mu\nu}(f, D)\left(1 + \sum_{j=1}^{N} |l_j(P; P_1, \dots, P_N)|\right). \end{aligned} \tag{73.9}$$

Since from (73.5) follows

$$|l_j(P; P_1, \dots, P_N)| \leqslant 1, \qquad j = 1, \dots, N, \tag{73.10}$$

we have

$$|f(P) - \pi_{\mu\nu}(P)| \leqslant (N + 1)\, d_{\mu\nu}(f, D). \tag{73.11}$$

The points $P_1, \dots, P_N$ can be determined for not too large values of μ and ν by numerical methods. On the other hand, we have

PROBLEM LXXXVIII. *What can be said about the distribution of* $P_1, \dots, P_N$, *satisfying the extremal condition* (73.4) *if* μ *and* ν *tend to infinity*?

Finally, a problem which needs no comment.

PROBLEM LXXXIX (V. T. Sós). *Do classes of functions* (*for instance, on the unit square*) *exist, for which approximation in the supremum norm by rational functions*

$$\frac{\pi_{\mu_1\nu_2}(x, y)}{\pi_{\mu_1\nu_2}(x, y)}$$

is essentially better than approximation by polynomials

$$\pi_{kl}(x, y),$$

if only

$$\frac{3}{4} \leqslant \frac{(k+1)(l+1)}{(\mu_1 + 1)(\nu_1 + 1) + (\mu_2 + 1)(\nu_2 + 1)} \leqslant 1\,?$$

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